

Convex and star-shaped sets associated with multivariate stable distributions

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Abstract

It is known that each symmetric stable distribution in \mathbb{R}^d is related to a norm on \mathbb{R}^d that makes \mathbb{R}^d embeddable in $L_p([0, 1])$. In case of a multivariate Cauchy distribution the unit ball in this norm is the polar set to a convex set in \mathbb{R}^d called a zonoid. This work interprets general stable laws using convex or star-shaped sets and exploits recent advances in convex geometry in order to come up with new probabilistic results for multivariate stable distributions. In particular, it provides expressions for moments of the Euclidean norm of a stable vector, mixed moments and various integrals of the density function. It is shown how to use geometric inequalities in order to bound important parameters of stable laws. Furthermore, covariation, regression and orthogonality concepts for stable laws acquire geometric interpretations. A similar collection of results is presented for one-sided stable laws.

Keywords: convex body; generalised function; Fourier transform; multivariate stable distribution; one-sided stable law; star body; spectral measure; support function; zonoid

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1 Introduction

Since P. Lévy it is well known that the characteristic function of a symmetric stable law in \mathbb{R}^d can be represented as the exponential of the norm as $\varphi(u) = e^{-\|u\|^p}$, where p is the characteristic exponent of the stable law. It is also well known [4, 20] that all norms which might appear in this representation make $(\mathbb{R}^d, \|\cdot\|)$ isometrically embeddable in the space $L_p([0, 1])$ if $p \geq 1$. The corresponding result holds also for all $p \in (0, 2]$, see [26, Lemma 6.4].

Each *norm* in \mathbb{R}^d gives rise to the corresponding *unit ball* F . In this paper we neglect the convexity property of the norm and use the term *norm*, where other works sometimes use the term *gauge function*. If the norm is convex, it can be realised as the support function $\|u\| = h(K, u)$ for the set K polar to the unit ball F , see Section 2. It is known from convex geometry (see [48] for the standard reference) that unit balls in spaces embeddable in $L_1([0, 1])$ are exactly polar sets to *zonoids*. Recall that zonoids appear as Hausdorff limits of zonotopes, i.e. finite Minkowski sums of segments. In application to stable laws with characteristic exponent $p = 1$, we have that $\varphi(u) = e^{-h(K, u)}$ is a characteristic function (necessarily corresponding to the multivariate Cauchy distribution) if and only if K is a zonoid. It is known that all planar centrally symmetric convex sets are zonoids, while this is no longer the case in dimensions 3 and more. This corresponds to a result of Ferguson [8], who showed that the dependency structure of symmetric *bivariate* Cauchy distributions can be described using *any* norm in \mathbb{R}^2 , while such representation is no longer possible for *all* norms in dimensions three and more.

It is explained in Section 3 that the correspondence between norms and stable laws can be extended to include all symmetric stable laws by representing their characteristic functions using (possibly non-convex) norms. The unit ball F in the corresponding norm is a star-shaped set called the star body associated with the stable law. Thus, each symmetric stable distribution is uniquely determined by a star body F and the value of the characteristic exponent α and the paper aims to show that this geometric interpretation is useful in the studies of multivariate stable laws. In particular, F is an ellipsoid if and only if the underlying distribution is sub-Gaussian. Section 4 shows that if the characteristic exponent p is at least one, then the norms are convex and the support function representation is also possible using the associated zonoid K being the polar set to F . This associated zonoid K is called L_p -zonoid, since K can be represented as the Hausdorff limit of a

power sum of segments, where the support function of K is the p -mean of the support functions of the summands.

While the general correspondence between zonoids and stable laws is well understood, this paper concentrates on further relationships between probabilistic aspects of stable laws and geometric properties of associated star-shaped and convex sets. The core of the paper begins in Section 5, where it is shown how to relate the value of the probability density function f of the symmetric stable law at zero to the volume of the associated star body F . It is shown how derivatives of f at the origin are related to further geometric properties of F , in particular to certain ellipsoids associated with F . It also provides an expression for the Rényi entropy of symmetric stable laws. Using geometric results on approximation of convex sets with ellipsoids, Section 5 ends up with a result that gives an estimate for the quality of approximation of a symmetric stable law with a sub-Gaussian one.

Section 6 uses the Fourier analysis for generalised functions together with the geometric representation of the characteristic function in order to compute a number of important characteristics of symmetric stable laws. These characteristics include the moments of the norm of a stable random vector, which previously were known only in the isotropic case, mixed moments of (possibly signed) powers of the coordinates, integrals of the density over subspaces, etc. This section also presents a number of inequalities for the moments and settles the equality cases. Finally, it clarifies a relationship between zonoids of stable laws and zonoids of random vectors studied in [44].

Section 7 deals with one-sided strictly stable laws supported by \mathbb{R}_+^d . It first establishes a geometric characterisation of stable laws for power sums, which fill the gap between the arithmetic addition and the coordinatewise maximum scheme for random vectors. Note that relationships between *max-stable* random vectors and convex sets have been explored in [43]. It is known that max-stable distributions with unit Fréchet marginals are exactly those having the cumulative distribution function $F(u^{-1}) = e^{-h(K,u)}$, where u^{-1} is the vector composed of the reciprocals of u and K is a *max-zonoid*, i.e. the expectation of a random crosspolytope. It is shown in Section 7 that stable laws for power sums (and also one-sided strictly stable laws) correspond to a new family of convex sets called $L_1(p)$ -zonoids. These sets appear as (set-valued) expectations of randomly rescaled ℓ_q -balls in \mathbb{R}^d , where q is reciprocal to p . Finally, it provides expressions for some moments of one-sided strictly stable laws.

The star bodies and zonoids associated with stable laws are determined by the spectral measures of stable laws. Section 8 shows that under quite general conditions, the spectral measures themselves admit a geometric interpretation as, e.g., surface area measures of further auxiliary convex sets called spectral bodies of stable laws. It shows how geometric inequalities can be used to relate volumes of the corresponding convex sets, and thereupon derive bounds for densities and moments of stable laws.

The geometric interpretation of the covariation is given in Section 9. It also discusses the regression problem for symmetric stable laws, in particular, the linearity property of multiple regression, which goes back to W. Blaschke's characterisation theorem for ellipsoids.

Section 10 describes several operations with associated star bodies and zonoids and their probabilistic meaning. Using recent approximation results from convex geometry, it is proved that each symmetric stable law can be obtained as the limit for sums of sub-Gaussian laws. It also discusses optimisation ideas, which appear, e.g. in optimising a portfolio whose components have jointly stable distribution.

Section 11 discusses the concept of James orthogonality for symmetric stable random variables, which is also extended to define orthogonality of symmetric stable random vectors.

This work attempts to highlight novel relationships between convex geometry and the theory of stable distributions. Further developments are surely possible by invoking other recent results on isotropic bodies, geometric concentration inequalities, or properties of convex sets in spaces of high (but finite) dimension. It is possible to apply the results to deal with finite-dimensional distributions of stable processes or work directly in a general infinite dimensional setting.

2 Star bodies and convex sets

A set F in \mathbb{R}^d is *star-shaped* if $[0, u] \subset F$ for each $u \in F$. A closed bounded set F is called a *star body* if for every $u \in F$ the interval $[0, u)$ is contained in the interior of F and the *Minkowski functional* (or the gauge function) of F defined by

$$\|u\|_F = \inf\{s \geq 0 : u \in sF\}$$

is a continuous function of $u \in \mathbb{R}^d$. The set F can be recovered from its Minkowski functional by

$$F = \{u : \|u\|_F \leq 1\},$$

while the *radial function*

$$\rho_F(u) = \|u\|_F^{-1}$$

provides the polar coordinate representation of the boundary of F for u from the unit *Euclidean sphere* \mathbb{S}^{d-1} . In the following we usually consider origin-symmetric star-shaped sets and call them *centred* in this case. If the star body F is centred and *convex*, then $\|u\|_F$ becomes a convex norm on \mathbb{R}^d .

The ℓ_p -ball in \mathbb{R}^d is defined by

$$B_p^d = \{x \in \mathbb{R}^d : \|x\|_p \leq 1\},$$

where $\|x\|_p = (|x_1|^p + \dots + |x_d|^p)^{1/p}$ for $p \neq 0$, i.e. $\|x\|_p = \|x\|_F$ with $F = B_p^d$.

If $p \in \mathbb{R} \setminus \{0\}$, the *p-star sum* of two star bodies F_1 and F_2 is defined by its Minkowski functional as

$$\|u\|_{F_1 \tilde{+}_p F_2} = (\|u\|_{F_1}^p + \|u\|_{F_2}^p)^{1/p}, \quad u \in \mathbb{R}^d. \quad (2.1)$$

This definition goes back to Firey [9] and was later investigated by Lutwak [33]. For $p = -1$ we obtain the radial sum, i.e. the radial function of the result is the sum of two radial functions of the summands. Extended by the limit for $p = -\infty$, the p -sum yields the union $F_1 \cup F_2$ and the intersection $F_1 \cap F_2$ for $p = \infty$. Note that (2.1) means that the Minkowski functional of $F_1 \tilde{+}_p F_2$ is proportional to the p -mean of the Minkowski functionals of F_1 and F_2 . See [19] for a comprehensive study of p -means of real numbers.

A convex set K in \mathbb{R}^d is called a *convex body* if K is compact and has non-empty interior. We usually use the letter F for star bodies (which are not necessarily convex) and K for convex bodies.

The *support function* of a bounded set A in \mathbb{R}^d is defined by

$$h(A, u) = \sup\{\langle x, u \rangle : x \in A\}, \quad u \in \mathbb{R}^d. \quad (2.2)$$

Clearly, $h(A, u)$ coincides with the support function of the convex hull of A . Any function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, which is sublinear, i.e. $f(cx) = cf(x)$ for all $c \geq 0$ and $f(x+y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}^d$, is a support function of a convex compact set, see [48, Th. 1.7.1].

The *polar set* to a convex body K is defined by

$$K^* = \{u : h(K, u) \leq 1\}. \quad (2.3)$$

The same definition applies if K is not necessarily convex. If K is convex, then $F = K^*$ is also convex and

$$\|u\|_F = h(K, u), \quad u \in \mathbb{R}^d,$$

i.e. the Minkowski functional of F is the support function of K .

The *Firey p -sum* of convex sets K_1 and K_2 that both contain the origin can be defined for $p \geq 1$ as the convex set $L = K_1 +_p K_2$ with the support function

$$h(L, u) = (h(K_1, u)^p + h(K_2, u)^p)^{1/p}, \quad (2.4)$$

see [10] and [32]. The Firey sum is closely related to the p -star sum (2.1), since

$$(K_1 +_p K_2)^* = K_1^* \tilde{+}_p K_2^*$$

for convex K_1 and K_2 that contain the origin. If $p = 1$, the Firey sum turns into the *Minkowski sum* defined as

$$K_1 + K_2 = \{x_1 + x_2 : x_1 \in K_1, x_2 \in K_2\}.$$

Then

$$h(K_1 + K_2, u) = h(K_1, u) + h(K_2, u), \quad u \in \mathbb{R}^d.$$

Further $\|x\|$ (without subscript) denotes the *Euclidean norm* of $x \in \mathbb{R}^d$. The (Euclidean) norm of a set K is defined as $\|K\| = \sup\{\|u\| : u \in K\}$. By $\text{Vol}_d(K)$ or $|K|$ we denote the d -dimensional *Lebesgue measure* of K . The volume of the unit ℓ_2 -ball (i.e. Euclidean ball) B in \mathbb{R}^d is denoted by

$$\kappa_d = |B| = \frac{\pi^{d/2}}{\Gamma(1 + \frac{d}{2})},$$

where Γ is the *Gamma function*. The same expression for κ_d is used also for all real $d > 0$.

A *random closed set* in \mathbb{R}^d is a random element in the space of closed sets equipped with the Fell topology and the corresponding Borel σ -algebra, see [42]. A random closed set X is said to be *compact* if X has a.s. compact realisations. If X is a random compact set in \mathbb{R}^d such that $\|X\|$ is

integrable, then $\mathbf{E} h(X, u)$ is the support function of a convex compact set called the (selection or Aumann) *expectation* of X and denoted by $\mathbf{E} X$, see [42, Sec. 2.1]. If X is a simple random convex set, i.e. X takes only a finite number of convex compact values K_1, \dots, K_n with probabilities p_1, \dots, p_n , then $\mathbf{E} X = p_1 K_1 + \dots + p_n K_n$. The expectation of a general X can be obtained by approximating X with simple random sets, see [42, Th. 2.1.21].

By applying (2.4) to simple random sets it is possible to define the *Firey p -expectation* $\mathbf{E}_p X$, $p \geq 1$, of a random compact set X such that $0 \in X$ almost surely and $\mathbf{E} \|X\|^p < \infty$. In particular,

$$h(\mathbf{E}_p X, u) = (\mathbf{E}[h(X, u)^p])^{1/p}$$

is the p -mean of $h(X, u)$ for $p \geq 1$, see also [11].

Sets that appear as finite Minkowski sums of segments are called zonotopes. *Zonoids* are limits of zonotopes in the Hausdorff metric, i.e. they can be represented as expectations of random segments. By changing the Minkowski sum to the Firey p -sum with $p \geq 1$ one obtains L_p -zonoids, which appear as limits for Firey p -sums of centred segments. This generalisation in the geometric context has been first mentioned in [17] and has been thoroughly investigated in [36]. It should be noted that L_p -zonoids are exactly those sets that appear as polar sets to the unit balls in spaces isometric to a d -dimensional subspace of $L_p([0, 1])$, see [24].

If X is a random closed set with almost surely star-shaped realisations (i.e. random star-shaped set), then the *p -star expectation* of X is the star-shaped set $F = \mathbf{E}_p^* X$ whose Minkowski functional is given by

$$\|u\|_F = (\mathbf{E} \|u\|_X^p)^{1/p}.$$

This expectation defines a star-shaped set for all $p \neq 0$. Note however that F is not necessarily a star body, since $\|u\|_F$ may be infinite. If X is a random convex body that contains the origin and satisfies $\mathbf{E} \|X\|^p < \infty$, then

$$\mathbf{E}_p X = \mathbf{E}_p^*(X^*), \quad p \geq 1. \quad (2.5)$$

3 Star bodies associated with $S\alpha S$ distributions

A random vector $\xi \in \mathbb{R}^d$ is called *symmetric α -stable* (notation $S\alpha S$) if ξ coincides in distribution with $-\xi$ and, for all $a, b > 0$,

$$a^{1/\alpha}\xi_1 + b^{1/\alpha}\xi_2 \stackrel{\mathcal{D}}{=} (a+b)^{1/\alpha}\xi,$$

where ξ_1, ξ_2 are independent copies of ξ , and $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution. The value of α is called the *characteristic exponent* of ξ . It is well known that ξ is normally distributed if and only if $\alpha = 2$.

Theorem 3.1 (see Th. 2.4.3 [47]). *A random vector ξ is $S\alpha S$ with $\alpha \in (0, 2)$ if and only if there exists a unique symmetric finite measure σ on the unit sphere \mathbb{S} in \mathbb{R}^d such that the characteristic function of ξ is given by*

$$\varphi_\xi(u) = \mathbf{E} e^{i\langle \xi, u \rangle} = \exp \left\{ - \int_{\mathbb{S}} |\langle u, z \rangle|^\alpha \sigma(dz) \right\}. \quad (3.1)$$

The measure σ is called the *spectral measure* of ξ . Representation (3.1) holds also for $\alpha = 2$, although the spectral measure is not necessarily unique in this case. Although the sphere \mathbb{S} can be defined with respect to any chosen (reference) norm in \mathbb{R}^d , in this paper we only use the Euclidean reference norm, so that $\mathbb{S} = \mathbb{S}^{d-1}$ is the Euclidean sphere in \mathbb{R}^d .

The expression in the exponential in the right hand side of (3.1) is an even homogeneous (of order α) function of u and so defines the Minkowski functional of a centred star body F as

$$\|u\|_F^\alpha = \int_{\mathbb{S}^{d-1}} |\langle u, z \rangle|^\alpha \sigma(dz), \quad (3.2)$$

so that

$$\varphi_\xi(u) = e^{-\|u\|_F^\alpha}, \quad u \in \mathbb{R}^d. \quad (3.3)$$

The star body F is called the *associated star body* of ξ . Since $\|u\|_F$ is finite, F always contains a neighbourhood of the origin. If σ is not concentrated on a great sub-sphere of \mathbb{S}^{d-1} (i.e. the intersection of \mathbb{S}^{d-1} with a $(d-1)$ -dimensional subspace), then ξ is called *full-dimensional*. In this case $\|u\|_F > 0$ for all $u \neq 0$.

The right-hand side of (3.2) is called the α -cosine transform of σ , which is studied also for all $\alpha > -1$, see [18], where it is shown that σ is unique if α is not an even integer. Although σ is not unique for $\alpha = 2$, the star body associated with the normal law is a unique ellipsoid.

In Section 4 we see that F is convex if $\alpha \in [1, 2]$. Example 4.6 describes $S\alpha S$ laws with $\alpha < 1$ whose associated star bodies are convex. The following simple examples deal with general $\alpha \in (0, 2]$.

Example 3.2 (Complete independence). If ξ is $S\alpha S$ with independent components, then

$$\varphi_\xi(u) = e^{-\|u\|_\alpha^\alpha},$$

i.e. the star body F associated with ξ is ℓ_α -ball B_α^d . This star body is not convex if $\alpha < 1$.

Example 3.3 (Complete dependence). If $\xi = (\xi_1, \dots, \xi_1)$ for $S\alpha S$ random variable ξ_1 , then

$$\varphi_\xi(u) = \exp \left\{ - \left| \sum u_i \right|^\alpha \right\}.$$

Note that ξ is not full-dimensional, so that $\rho_F(u) = \|u\|_F^{-1} = |\sum u_i|^{-1}$ is the radial function of an unbounded star body F . The corresponding polar set F^* is the segment with end-points $\pm(1, \dots, 1)$.

The associated star body of $S\alpha S$ random vector ξ can be obtained as $F = c^{-1/\alpha} \mathbf{E}_\alpha^* Y_\eta$, i.e. F is the star expectation of

$$Y_\eta = \{x : |\langle x, \eta \rangle| \leq 1\},$$

where c is the total mass of σ and η is distributed according to $c^{-1}\sigma$. For this, it suffices to note that $|\langle u, z \rangle| = \|u\|_{Y_z}$ where Y_z is the polar set to $[-z, z]$. It is often useful to redefine the spectral measure σ to be a probability measure on the whole \mathbb{R}^d . Then no constant c is needed, so that $F = \mathbf{E}_\alpha^* Y_\eta$, where η is distributed in \mathbb{R}^d according to σ . Note that it is always possible to extend σ to be a square integrable on \mathbb{R}^d , so that the integral (3.2) exists for all $\alpha \in (0, 2]$.

A centred convex body F in \mathbb{R}^d is called an L_p -ball if it is the unit ball of a d -dimensional subspace of $L_p([0, 1])$. Denote by \mathcal{L}_p the family of L_p -balls. It is known that $F \in \mathcal{L}_p$ if and only if

$$\|u\|_F^p = \int_{\mathbb{S}^{d-1}} |\langle u, z \rangle|^p \mu(dz) \quad (3.4)$$

for a finite measure μ on \mathbb{S}^{d-1} , see [18, Lemma 4.8] for $p \geq 1$ and [26, Lemma 6.4] for general $p > 0$. Note that (3.4) is called the Blaschke-Lévy representation of the norm, which is discussed in detail in [25]. It should be noted that $\exp\{-\|u\|_F^p\}$ is positive definite for $p \in (0, 2]$ if and only if $F \in \mathcal{L}_p$, see [23, 24] for a survey of related results. By comparing (3.4) with Theorem 3.1 and symmetrising, if necessary, the measure μ , we see that \mathcal{L}_α is exactly the family of associated star bodies of $S\alpha S$ laws. In the following we often switch between the letters α and p , since the former is common in the literature on stable laws, while the latter is typical in convex geometry and functional analysis.

It is shown in [26, Cor. 6.7] that if F is an L_p -ball with $p \in (0, 2]$, then F is also an L_r -ball for each $r \in (0, p)$. It is instructive to provide a probabilistic proof of this fact.

Theorem 3.4. *If F is an L_p -ball for $p \in (0, 2]$, then F is also an L_r -ball for all $r \in (0, p]$.*

Proof. Consider an $S\alpha S$ random vector ξ with $\alpha = p$ and the associated star body F . Let ζ be a non-negative stable random variable with $\beta \in (0, 1)$. Then the characteristic function of $\xi' = \zeta^{1/\alpha}\xi$ is given by

$$\mathbf{E} e^{i\langle \xi', u \rangle} = e^{-\|u\|_F^{\alpha\beta}}.$$

Thus, F is the associated star body of the symmetric stable ξ' with the characteristic exponent $\alpha\beta$, so that F is an L_r -ball for $r = \alpha\beta = p\beta < p$. Note that ξ and ξ' share the same associated star body F . \square

4 Zonoids and $S\alpha S$ laws with $\alpha \in [1, 2]$

If $\alpha \in [1, 2]$, it is possible to arrive at a dual interpretation of the characteristic function (3.1) by noticing that $|\langle u, x \rangle|$ is the support function of the segment $[-x, x]$, so that

$$\int_{\mathbb{S}^{d-1}} |\langle u, z \rangle|^\alpha \sigma(dz) = \int_{\mathbb{S}^{d-1}} h([-z, z], u)^\alpha \sigma(dz) = h(K, u)^\alpha, \quad (4.1)$$

where

$$K = \sigma(\mathbb{S}^{d-1})^{1/\alpha} \mathbf{E}_\alpha[-\eta, \eta]$$

is the rescaled Firey α -expectation of the random set $X = [-\eta, \eta]$ and η is a random vector with values in \mathbb{S}^{d-1} distributed according to the normalised spectral measure σ . The Minkowski inequality implies that $h(K, u)$ is indeed a support function of a convex set. If $\alpha = 1$, then K is called a zonoid, see [48, Sec. 3.5]. Note that representation (4.1) appears already in [4] and [20] in view of its relationship to stable distributions and negative definite functions on one hand and L_p -balls on the other one.

Definition 4.1. Let σ be a finite measure on \mathbb{S}^{d-1} . A convex set K in \mathbb{R}^d is called L_p -zonoid with $p \geq 1$ and spectral measure σ if $K = c^{1/p} \mathbf{E}_p[-\eta, \eta]$, where c is the total mass of σ and η is distributed according to $c^{-1}\sigma$.

It is obvious that L_1 -zonoids are conventional zonoids. If σ is a p -integrable probability measure on \mathbb{R}^d , then the L_p -zonoid can be defined as $\mathbf{E}_p[-\eta, \eta]$ where η has distribution σ . The following result now becomes an easy corollary from Theorem 3.1.

Theorem 4.2. A random vector ξ is $S\alpha S$ with $\alpha \in [1, 2]$ if and only if there exists a unique centred L_α -zonoid K such that the characteristic function of ξ is given by

$$\varphi_\xi(u) = e^{-h(K, u)^\alpha}, \quad u \in \mathbb{R}^d. \quad (4.2)$$

The set K from Theorem 4.2 is said to be the *associated zonoid* of ξ . The corresponding polar set $F = K^*$ is convex and becomes the associated star body of ξ . It is well known that all centred convex compact sets on the plane are L_1 -zonoids (i.e. classical zonoids), while this no longer holds in dimensions 3 and more. It follows immediately from Theorem 3.4 that the family of L_p -zonoids becomes richer if $p \in [1, 2]$ decreases.

The following result provides a further interpretation of the well-known fact saying that the exponentials of support functions of zonoids are positive definite, see [48, p. 194].

Corollary 4.3. The function $\varphi(u) = e^{-h(K, u)^\alpha}$, $u \in \mathbb{R}^d$, with $\alpha \in [1, 2]$ and a centred convex body $K \subset \mathbb{R}^d$ is a characteristic function if and only if K is L_α -zonoid. In this case φ is necessarily the characteristic function of $S\alpha S$ random vector.

A measure σ on \mathbb{S}^{d-1} is called *isotropic* if the L_2 -zonoid with spectral measure σ is a centred Euclidean ball, see [36, 41]. In other words, if an isotropic σ is taken as the spectral measure of a Gaussian random vector,

then this Gaussian vector has i.i.d. coordinates. The two most common examples are the uniform measure on \mathbb{S}^{d-1} and the cross measure having atoms of equal weights at $\pm e_i$ for the canonical basis e_1, \dots, e_d . Note that the isotropy of σ does not mean that the corresponding $S\alpha S$ vectors (with α not necessarily equal 2) has a Euclidean ball as its associated star body.

Example 4.4 (Independent/completely dependent components). The components of $S\alpha S$ vector ξ with $\alpha \in [1, 2]$ are independent if and only if its associated zonoid K is a rescaled ℓ_α -ball, i.e.

$$K = \{(a_1 x_1, \dots, a_d x_d) : x \in B_\alpha^d\}$$

for $a_1, \dots, a_d \in \mathbb{R}$. If some of the a_i 's vanish, then ξ is no longer full-dimensional. Thus, an ℓ_q -ball is L_r -zonoid for all $r \in [1, p]$ with p being reciprocal to q .

Furthermore, $\xi = (a_1 \xi_1, \dots, a_d \xi_1)$ for $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ and so has completely dependent components if and only if K is the segment with end-points $\pm a$. In this case ξ is not full-dimensional for each a and $d \geq 2$.

Example 4.5 (Ellipsoids and sub-Gaussian laws). The family of full-dimensional L_2 -zonoids is the family of centred *ellipsoids* in \mathbb{R}^d , that also correspond uniquely to non-degenerate Gaussian laws on \mathbb{R}^d . Thus ellipsoids are also L_p -zonoids for any $p \in [1, 2]$. Since polar sets to ellipsoids are again ellipsoids, ellipsoids are also L_p -balls for each $p \in (0, 2]$. Ellipsoids do not have a unique spectral measure for $\alpha = 2$. However, if an ellipsoid is represented as an L_p -zonoid with $p \in [1, 2)$ or an L_p -ball with $p \in (0, 2)$, then its spectral measure is unique. The corresponding $S\alpha S$ random vector is said to have a *sub-Gaussian* distribution, see [47, Sec. 2.5].

An *elliptical norm* is determined by a positive definite symmetric matrix C , so that $\|u\|_E = \langle Cu, u \rangle$ for the corresponding centred ellipsoid E . A simple quadratic optimisation argument yields that

$$h(E, u) = \sqrt{\langle C^{-1}u, u \rangle},$$

see, e.g., [21].

Example 4.6 (Sub-stable laws). The distribution of ξ' from the proof of Theorem 3.4 is called *sub-stable*. If ξ is $S\alpha S$ with $\alpha \in [1, 2)$ and the associated zonoid K , then ξ' is stable with the characteristic exponent $\alpha' = \alpha\beta$ and

$$\mathbf{E} e^{i\langle \xi', u \rangle} = e^{-h(K, u)^{\alpha\beta}} = e^{-\|u\|_{K^*}^{\alpha\beta}}.$$

In this case the star body associated with ξ' is convex and is equal to the polar set to K . In particular, this holds for all sub-Gaussian distributions whose associated star bodies are ellipsoids for each $\alpha \in (0, 2)$.

Theorem 4.7. *Each L_p -zonoid with $p > 1$ and spectral measure which is not concentrated on a great sub-sphere of \mathbb{S}^{d-1} is strictly convex, i.e. its support function is differentiable at every point.*

Proof. If $p > 1$, then $|\langle u, v \rangle|^p$ is a differentiable function of u , so its integral is also differentiable. Since σ is full-dimensional, the integral with respect to σ does not vanish, so that its $\frac{1}{p}$ th power is also differentiable. The equivalence of strict convexity and differentiability properties is explained in [48, Cor. 1.7.3]. \square

The strict convexity of K means that for each $u \in \mathbb{R}^d$ the *support set*

$$T(K, u) = \{y \in K : \langle y, u \rangle = h(K, u)\} \quad (4.3)$$

is a singleton $\{x\}$ and the gradient of $h(K, u)$ equals x . Theorem 4.7 implies that polytopes cannot be L_p -zonoids for $p > 1$, so that the approximation by polytopes (often used in the studies of zonoids) is no longer useful for L_p -zonoids with $p > 1$.

5 Symmetric stable densities

5.1 Value of the density at the origin

Consider $S\alpha S$ random vector ξ with $\alpha \in (0, 2]$ and the characteristic function given by (3.3). It is useful to interpret this characteristic function as

$$\varphi_\xi(u) = e^{-\|u\|_F^\alpha} = \mathbf{P}\{\zeta \geq \|u\|_F\} = \mathbf{E} \mathbb{I}_{\zeta \geq \|u\|_F} = \mathbf{E} \mathbb{I}_{u \in \zeta F}, \quad (5.1)$$

where ζ is a non-negative random variable with $\mathbf{P}\{\zeta \geq x\} = e^{-x^\alpha}$ for $x > 0$, so that

$$\mathbf{E} \zeta^\lambda = \Gamma(1 + \lambda/\alpha), \quad \lambda > -\alpha. \quad (5.2)$$

The inversion formula for the Fourier transform yields the following expression for the probability density function f of ξ

$$\begin{aligned} (2\pi)^d f(x) &= \int_{\mathbb{R}^d} e^{-i\langle u, x \rangle} \varphi_\xi(u) du = \mathbf{E} \int_{\mathbb{R}^d} e^{-i\langle u, x \rangle} \mathbb{1}_{u \in \zeta F} du \\ &= \mathbf{E} \int_{\zeta F} e^{i\langle u, x \rangle} du. \end{aligned} \quad (5.3)$$

Note that we have used the fact that F is centred. Since f is the expectation of the characteristic function of the uniform law on ζF , the bounds on this characteristic function (see, e.g., [29, Th. 1]) can be used to derive bounds for f .

By substituting $x = 0$ in (5.3) we obtain

$$f(0) = \frac{1}{(2\pi)^d} \Gamma(1 + \frac{d}{\alpha}) |F|. \quad (5.4)$$

Recall that the volumes of F and its polar set $K = F^*$ (in case $\alpha \geq 1$) are related by the Blaschke-Santaló inequality as

$$|F| \cdot |K| \leq \kappa_d^2$$

with the equality if and only if F is an ellipsoid, i.e. ξ is sub-Gaussian.

If the spectral measure σ is isotropic with the L_2 -zonoid being the unit Euclidean ball and $\alpha \geq 1$, then it is possible to apply the results from [36] in order to bound the volume of F as

$$\omega_d(2)/c_\alpha \leq |F| \leq \omega_d(\alpha), \quad (5.5)$$

where

$$\omega_d(\alpha) = 2^d \frac{\Gamma(1 + \frac{1}{\alpha})}{\Gamma(1 + \frac{d}{\alpha})}, \quad c_\alpha^{\alpha/d} = \frac{\Gamma(1 + \frac{d}{2})}{\Gamma(1 + \frac{1}{2})} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{d+\alpha}{2})}.$$

If $\alpha \in [1, 2)$, then the equality on the left in (5.5) is achieved if σ is a suitably normalised Lebesgue measure on \mathbb{S}^{d-1} , while the equality on the right holds if σ is concentrated on $\pm e_1, \dots, \pm e_d$.

5.2 Derivatives at the origin

Since $\varphi_\xi(u)$ multiplied by a product of powers of the coordinates of u is integrable, representation (5.3) implies that f is infinitely differentiable. Its

derivatives at the origin are given by

$$(2\pi)^d \frac{\partial^{2m} f}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} \Big|_{x=0} = (-1)^m \Gamma(1 + \frac{2m+d}{\alpha}) \int_F v_1^{k_1} \cdots v_d^{k_d} dv,$$

where $2m = k_1 + \cdots + k_d$. The central symmetry of F implies that the partial derivatives of odd orders vanish. By combining these partial derivatives (with $m = 1$) we arrive at the following expression

$$\left(\sum_{i=1}^d w_i \frac{\partial}{\partial x_i} \right)^2 f \Big|_{x=0} = - \frac{1}{(2\pi)^d} \Gamma(1 + \frac{2+d}{\alpha}) \int_F \langle w, v \rangle^2 dv,$$

where $w = (w_1, \dots, w_d)$. The integral in the right-hand side can be written as $\|w\|_E^2$ where E is an ellipsoid in \mathbb{R}^d called (for a convex F) the *Binet ellipsoid* of F . This ellipsoid is homothetic to the Legendre ellipsoid of F , which shares the moments of inertia with F , see [41]. Results from [41] can be used in order to bound the integral of $\langle w, v \rangle^2$ over F .

Note that

$$\sum_{i=1}^d w_i^2 \frac{\partial^2 f}{\partial x_i^2} \Big|_{x=0} = - \frac{1}{(2\pi)^d} \Gamma(1 + \frac{d+2}{\alpha}) \int_F \sum_{i=1}^d w_i^2 v_i^2 dv,$$

where $(w_1^2 x_1^2 + \cdots + w_d^2 x_d^2)$ defines an elliptic norm of x with the unit ball being the centred ellipsoid E with semi-axes $w_1^{-1}, \dots, w_d^{-1}$. Corollary 2.2a of [41] yields that

$$\int_F (w_1^2 x_1^2 + \cdots + w_d^2 x_d^2) dx \geq \frac{d}{d+2} |F|^{1+\frac{2}{d}} (w_1 \cdots w_d)^{2/d} \kappa_d^{-2/d},$$

and so provides an upper bound for the weighted sum of the second derivatives of the density f of ξ at the origin as

$$\sum_{i=1}^d w_i^2 \frac{\partial^2 f}{\partial x_i^2} \Big|_{x=0} \leq - \frac{4\pi d}{d+2} \frac{\Gamma(1 + \frac{d+2}{\alpha}) \Gamma(1 + \frac{d}{2})^{2/d}}{\Gamma(1 + \frac{d}{\alpha})^{1+2/d}} (w_1 \cdots w_d)^{2/d} f(0)^{1+2/d},$$

with the equality attained if F is a dilate of the ellipsoid E_w , i.e. for the corresponding sub-Gaussian law.

5.3 Expectation of integrable functions

Integrating (5.3) leads to the following expression

$$\mathbf{E} g(\xi) = \frac{1}{(2\pi)^d} \mathbf{E} \left[\int_{\zeta F} \hat{g}(-v) dv \right], \quad (5.6)$$

where \hat{g} is the Fourier transform of an integrable function g . If g is the Fourier transform of a measure μ , then $\mathbf{E} g(\xi) = \mathbf{E} \mu(\zeta F)$. For example, $\mathbf{E} \exp\{-\|\xi\|^2/2\}$ equals the expected standard Gaussian content of ζF . If g is the indicator of the Euclidean ball B_r of radius r centred at the origin, then

$$\hat{g}(u) = (2r\pi/\|u\|)^{d/2} J_{d/2}(r\|u\|),$$

where $J_{d/2}$ is the Bessel function. Therefore

$$\mathbf{P}\{\|\xi\| \leq r\} = \left(\frac{r}{2\pi}\right)^{d/2} \int_F \|v\|^{-d/2} \mathbf{E}[\zeta^{d/2} J_{d/2}(r\zeta\|v\|)] dv.$$

It is also possible to choose $g(\xi)$ to be the product of functions of individual coordinates of ξ , i.e.

$$g(\xi) = \prod_{i=1}^d g_i(\xi_i).$$

For instance, if $g_i(x_i) = \mathbb{I}_{[-a_i, a_i]}(x_i)$, $i = 1, \dots, d$, then

$$\mathbf{E} \left[\prod_{i=1}^d g_i(\xi_i) \right] = \mathbf{P}\{\xi \in \times_{i=1}^d [-a_i, a_i]\} = \pi^{-d} \int_F \mathbf{E} \prod_{i=1}^d \frac{\sin(a_i v_i \zeta)}{v_i} dv.$$

The same argument with the Laplace density $g_i(x_i) = \frac{\lambda_i}{2} e^{-\lambda_i |x_i|}$ yields that

$$\mathbf{E} \exp \left\{ - \sum \lambda_i |\xi_i| \right\} = \pi^{-d} \int_F \mathbf{E} \left[\prod_{i=1}^d \frac{\zeta \lambda_i}{\zeta^2 v_i^2 + \lambda_i^2} \right] dv.$$

Note that in all these cases the dependency structure is expressed by the set F which determines the integration domain, while the value of α influences the integrand which is the expectation of a certain function of ζ .

5.4 Rényi entropy and related integrals

Another instance of (5.6) appears if g is itself the density of $S\alpha'S$ law with associated star body F' . Then

$$\mathbf{E} g(\xi) = \frac{1}{(2\pi)^d} \mathbf{E} |\zeta F \cap \zeta' F'|, \quad (5.7)$$

where $\mathbf{P}\{\zeta' > x\} = e^{-x^{\alpha'}}$ and ζ' is independent of ζ .

Theorem 5.1. *If ξ is $S\alpha S$ with associated star body F and $\alpha \in (0, 2]$, then, for all $c \neq 0$, the density f of ξ satisfies*

$$\int_{\mathbb{R}^d} f(cx) f(x) dx = (1 + c^\alpha)^{-d/\alpha} f(0).$$

Proof. Apply (5.7) with $g(x) = c^d f(cx)$ and $\alpha = \alpha'$, so that $F' = cF$. Then

$$\begin{aligned} \mathbf{E} g(\xi) &= c^d \int_{\mathbb{R}^d} f(x) f(cx) dx = \frac{1}{(2\pi)^d} |F| \mathbf{E} (\min(\zeta, c\zeta'))^d \\ &= \frac{1}{(2\pi)^d} \frac{\Gamma(1 + \frac{d}{\alpha})}{(1 + c^{-\alpha})^{d/\alpha}} |F|. \end{aligned}$$

Then note that $|F|$ is related to $f(0)$ by (5.4). □

By choosing $c = 1$ we see that the density of each $S\alpha S$ law satisfies

$$\int_{\mathbb{R}^d} f(x)^2 dx = 2^{-d/\alpha} f(0).$$

The left-hand side can be recognised as the inverse to the 2-Rényi entropy power of ξ .

5.5 Probability metric and distance to sub-Gaussian law

Some useful *probability metrics* are defined using logarithms of characteristic functions of random vectors. Extending the definition of the distance between two random variables from [52, Ex. I.1.15] for the multivariate case, it is possible to calculate the distance between two $S\alpha S$ vectors ξ' and ξ'' with

the same characteristic exponent $\alpha \in [1, 2]$ and the associated zonoids K_1 and K_2 as

$$\begin{aligned} \mathbf{m}_\alpha(\xi', \xi'') &= \sup\{\|u\|^{-\alpha} |\log \mathbf{E} e^{i\langle u, \xi' \rangle} - \log \mathbf{E} e^{i\langle u, \xi'' \rangle}| : u \in \mathbb{R}^d\} \\ &= \sup\{|h(K_1, u)^\alpha - h(K_2, u)^\alpha| : u \in \mathbb{S}^{d-1}\}. \end{aligned}$$

If $\alpha = 1$, the right-hand side becomes the Hausdorff distance between K_1 and K_2 .

Theorem 5.2. *For each $S\alpha S$ vector ξ with $\alpha \in [1, 2]$ in \mathbb{R}^d and the associated zonoid K there exists a sub-Gaussian $S\alpha S$ vector η such that $\mathbf{m}_\alpha(\xi, \eta) \leq (d^{\alpha/2} - 1)\|K\|^\alpha$.*

Proof. For each centred convex body K in \mathbb{R}^d there exists a centred ellipsoid E (called the *John ellipsoid*) such that $E \subset K \subset \sqrt{d}E$, see e.g. [12, Th. 4.2.12]. Then it suffices to note that

$$|h(K, u)^\alpha - h(E, u)^\alpha| \leq |d^{\alpha/2}h(E, u)^\alpha - h(E, u)^\alpha| \leq h(E, u)^\alpha(d^{\alpha/2} - 1)$$

and use the fact that $h(E, u) \leq \|E\| \leq \|K\|$ for all $u \in \mathbb{S}^{d-1}$. \square

Results from Section 6 can be used to related moments of ξ and η . For instance, Theorem 6.1 implies that $\mathbf{E}\|\xi\|^\lambda / \mathbf{E}\|\eta\|^\lambda \in [1, d^{\lambda/2}]$.

6 Homogeneous functions of $S\alpha S$ laws

6.1 Moments of the norm

If g is a homogeneous function, i.e. $g(cx) = c^\lambda g(x)$ for all $x \in \mathbb{R}^d$ and $c > 0$, and so is not integrable over \mathbb{R}^d , then one can interpret its Fourier transform using generalised functions. We refer to [14] for the thorough account of generalised functions and their Fourier transforms. The left-hand side of (5.6) for not necessarily integrable g can be interpreted as the action of g on the density f of ξ (denoted (g, f)), while the right-hand side as the action of the Fourier transform \hat{g} of g on φ_ξ , i.e.

$$(g, f) = \frac{1}{(2\pi)^d}(\hat{g}, \varphi_\xi)$$

is Parseval's identity. Since φ_ξ given by (5.1) is not necessarily infinitely differentiable, the action of a generalised function on it should be interpreted as limits if the action of \hat{g} does not involve differentiation.

Theorem 6.1. *If ξ is $S\alpha S$ and $\lambda \in (-d, \alpha)$, then*

$$\mathbf{E} \|\xi\|^\lambda = \frac{2^{\lambda-1}}{\pi^{d/2}} \Gamma\left(\frac{d+\lambda}{2}\right) \frac{\Gamma(1-\frac{\lambda}{\alpha})}{\Gamma(1-\frac{\lambda}{2})} \int_{\mathbb{S}^{d-1}} \|u\|_F^\lambda du. \quad (6.1)$$

Proof. Consider (5.6) for $g(x) = \|x\|^\lambda = r^\lambda$. Using the expression for the Fourier transform of g (see [14, Sec. II.3.3]) one arrives at

$$\mathbf{E} \|\xi\|^\lambda = \frac{2^\lambda}{\pi^{d/2}} \frac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(-\frac{\lambda}{2})} \left(r^{-\lambda-d}, \mathbf{E} \mathbb{I}_{\zeta F} \right),$$

where $(r^{-\lambda-d}, \psi)$ denotes the action of the generalised function $r^{-\lambda-d}$ on the test function ψ . If $0 < \lambda < \alpha$, then it is possible to use the regularisation for $r^{-\lambda-d}$ (see [14, Sec. I.3.9]) to obtain

$$\left(r^{-\lambda-d}, \mathbf{E} \mathbb{I}_{\zeta F} \right) = \mathbf{E}(\zeta^{-\lambda}) \omega_d \int_0^\infty t^{-\lambda-1} (S_F(t) - 1) dt,$$

where ω_d is the surface area of the unit sphere in \mathbb{R}^d and $S_F(t)$ is the ratio of the surface areas of $S(t) \cap F$ and the sphere $S(t)$ of radius t . Then

$$\begin{aligned} \omega_d \int_0^\infty t^{-\lambda-1} (1 - S_F(t)) dt &= \int_{\mathbb{S}^{d-1}} \int_0^\infty t^{-\lambda-1} (1 - \mathbb{I}_{ut \in F}) dt du \\ &= \int_{\mathbb{S}^{d-1}} \int_{\|u\|_F^{-1}}^\infty t^{-\lambda-1} dt du, \end{aligned}$$

which, together with the expression (5.2) for the moment of ζ , proves (6.1) for $\lambda > 0$.

If $\lambda \in (-d, 0)$, then no regularisation is needed, so that

$$\mathbf{E} \|\xi\|^\lambda = \frac{2^\lambda}{\pi^{d/2}} \frac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(-\frac{\lambda}{2})} \Gamma\left(1 - \frac{\lambda}{\alpha}\right) \int_F \|u\|^{-\lambda-d} du.$$

Then (6.1) is obtained by passing to polar coordinates and using the fact that $\Gamma(1 - \lambda/2) = (-\lambda/2)\Gamma(-\lambda/2)$. A direct check shows that (6.1) holds also for $\lambda = 0$. \square

Remark 6.2. An alternative proof of Theorem 6.1 can be carried over using the *plane-wave expansion* of the Euclidean norm

$$\|x\|^\lambda = \frac{1}{2\pi^{(d-1)/2}} \frac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{1+\lambda}{2})} \int_{\mathbb{S}^{d-1}} |\langle u, x \rangle|^\lambda du,$$

see [14, Sec. 3.10] and the expression for the moments of $|\langle u, \xi \rangle|$, see Theorem 6.16.

Example 6.3 (Isotropic law). Assume that ξ is *isotropic*, i.e. $\|u\|_F = \sigma\|u\|$ for all u and $F = B_{\sigma^{-1}}$ is the ball of radius σ^{-1} . Then (6.1) and the expression for the surface area ω_d of the unit sphere imply that

$$\mathbf{E} \|\xi\|^\lambda = (2\sigma)^\lambda \frac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{d}{2})} \frac{\Gamma(1 - \frac{\lambda}{\alpha})}{\Gamma(1 - \frac{\lambda}{2})}, \quad \lambda \in (-d, \alpha),$$

which is a well-known formula, see, e.g., [51, Eq. (7.5.9)].

If it is difficult to integrate $\|u\|_F^\lambda$ over the unit sphere, it is possible to use trivial bounds $R^{-1} \leq \|u\|_F \leq r^{-1}$, where R and r are the radii of the circumscribed and the inscribed balls to F . The following lower bound is sharper for $\lambda > 0$.

Corollary 6.4. *In the setting of Theorem 6.1 with $\lambda \in (0, \alpha)$, we have*

$$\mathbf{E} \|\xi\|^\lambda \geq 2^\lambda \frac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{d}{2})} \frac{\Gamma(1 - \frac{\lambda}{\alpha})}{\Gamma(1 - \frac{\lambda}{2})} \left(\frac{\kappa_d}{|F|} \right)^{\lambda/d} \quad (6.2)$$

with the equality if and only if F is a Euclidean ball.

Proof. The expression

$$\tilde{V}_{-\lambda}(L, F) = \frac{1}{d} \int_{\mathbb{S}^{d-1}} \|u\|_L^{-d-\lambda} \|u\|_F^\lambda du$$

is called the *dual mixed volume* of star bodies K and L (note that the original definition [33] is written for the radial functions of L and F). Now it suffices to apply the dual mixed volume inequality (see [33] and [35, (2.4)])

$$\tilde{V}_{-\lambda}(L, F)^d \geq |L|^{d+\lambda} |F|^{-\lambda}$$

with L being the unit Euclidean ball. □

Note that the right-hand side of (6.2) equals $\mathbf{E} \|\eta\|^\lambda$, where η is an isotropic $S\alpha S$ random vector with the associated star body being the Euclidean ball of the same volume as F .

Example 6.5 (Multivariate normal and sub-Gaussian distributions). If ξ has a multivariate normal distribution with covariance matrix C , then F is the ellipsoid E with $\|u\|_E^2 = \frac{1}{2}\langle Cu, u \rangle$ and (6.1) implies

$$\mathbf{E} \|\xi\|^\lambda = \frac{2^{\lambda/2-1}}{\pi^{d/2}} \Gamma\left(\frac{d+\lambda}{2}\right) \int_{\mathbb{S}^{d-1}} \langle Cu, u \rangle^{\lambda/2} du \quad (6.3)$$

for $\lambda \in (-d, 2)$. By passing to the limit, the formula holds also for $\lambda = 2$. The integral retains its value for ξ having a sub-Gaussian distribution with the same associated star body F . Thus, the ratio of the moments of the norm for a normal vector and the corresponding $S\alpha S$ sub-Gaussian vector depends only on α , dimension and the order of the moment.

If $\sigma_1^2, \dots, \sigma_d^2$ are the eigenvalues of C , then $F = E$ has semi-axes $\sqrt{2}/\sigma_i$, $i = 1, \dots, d$, whence $\kappa_d/|F|$ equals $2^{-d/2} \prod \sigma_i$ and Corollary 6.4 yields that

$$\mathbf{E} \|\xi\|^\lambda \geq 2^{\lambda/2} \frac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{d}{2})} \frac{\Gamma(1 - \frac{\lambda}{\alpha})}{\Gamma(1 - \frac{\lambda}{2})} \left(\prod_{i=1}^d \sigma_i \right)^{\lambda/d} \quad (6.4)$$

with the equality if and only if $\sigma_1 = \dots = \sigma_d$. In particular, if C is diagonal, then

$$\mathbf{E}(\xi_1^2 + \dots + \xi_d^2)^{\lambda/2} \geq \frac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{d}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1+\lambda}{2})} \prod_{i=1}^d \mathbf{E} |\xi_i|^\lambda$$

with the equality if and only if $\mathbf{E} |\xi_i|^\lambda$ does not depend on i . For this, we have used (6.4) and the fact that

$$\mathbf{E} |\xi_i|^\lambda = 2^{\lambda/2} \frac{\Gamma(\frac{1+\lambda}{2})}{\Gamma(\frac{1}{2})} \frac{\Gamma(1 - \frac{\lambda}{\alpha})}{\Gamma(1 - \frac{\lambda}{2})} \sigma_i^\lambda.$$

Example 6.6 ($S\alpha S$ vectors with i.i.d. components). Let ξ be $S\alpha S$ with the associated star body being ℓ_α -ball with $\alpha \in (0, 2]$, so that its coordinates ξ_1, \dots, ξ_d are i.i.d. $S\alpha S$ random variables, see Example 3.2. The formula for the volume of the ℓ_p -ball from [45, p. 11] and (6.2) imply that

$$\mathbf{E}(\xi_1^2 + \dots + \xi_d^2)^{\lambda/2} \geq 2^{-\lambda} \frac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{1+\lambda}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})} \frac{\Gamma(1 + \frac{d}{\alpha})^{\lambda/d}}{\Gamma(1 + \frac{1}{\alpha})^\lambda} \kappa_d^{\lambda/d} \mathbf{E} |\xi_1|^\lambda$$

for all $\lambda \in (0, \alpha)$. For the opposite inequality note that the largest Euclidean ball inscribed in B_α^d has radius $d^{\frac{1}{2}-\frac{1}{\alpha}}$, whence

$$\mathbf{E}(\xi_1^2 + \dots + \xi_d^2)^{\lambda/2} \leq \frac{1}{\sqrt{\pi} \Gamma(\frac{d}{2})} \frac{\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{1+\lambda}{2})} d^{\frac{1}{\alpha}-\frac{1}{2}} \mathbf{E} |\xi_1|^\lambda$$

for $\lambda \in (0, \alpha)$. The both inequalities turn into an equality for $\alpha = 2$ and any $\lambda \in (0, 2]$ and yield the well-known moments of the chi-square distribution with d degrees of freedom.

Using bounds for the average values of norms on the unit sphere from [31], it is possible to relate moments of different orders.

Corollary 6.7. *Let ξ be $S\alpha S$ with $\alpha \in (1, 2]$ and the associated star body F . Let b be the radius of the largest centred Euclidean ball inscribed in F . Then for all $\lambda \in [1, \alpha)$*

$$a_\lambda \max \left(M_1, \frac{c_1 b \sqrt{\lambda}}{\sqrt{d}} \right)^\lambda \leq \mathbf{E} \|\xi\|^\lambda \leq a_\lambda \max \left(2M_1, \frac{c_2 b \sqrt{\lambda}}{\sqrt{d}} \right)^\lambda,$$

where c_1 and c_2 are absolute constants,

$$M_1 = \frac{\pi^{(d+1)/2}}{\Gamma(\frac{d+1}{2})\Gamma(1 - \frac{1}{\alpha})} \mathbf{E} \|\xi\| = \int_{\mathbb{S}^{d-1}} \|u\|_F du.$$

and

$$a_\lambda = \frac{2^{\lambda-1} \Gamma(\frac{d+\lambda}{2}) \Gamma(1 - \frac{\lambda}{\alpha})}{\pi^{d/2} \Gamma(1 - \frac{\lambda}{2})}.$$

Using [26, Lemma 3.6] for the Fourier transform of the power of the ℓ_p -norm $\|x\|_p^\lambda$ it is possible to arrive at the following expression

$$\mathbf{E} \|\xi\|_p^\lambda = \frac{1}{(2\pi)^d} \frac{p \Gamma(1 - \frac{\lambda}{\alpha})}{\Gamma(-\frac{\lambda}{p})} \int_F \int_0^\infty s^{d+\lambda-1} \prod_{i=1}^d \gamma_p(sv_i) ds dv,$$

where γ_p is the Fourier transform of the function $e^{-|x|^p}$, $x \in \mathbb{R}$. It is valid for $\lambda \in (-d, 0)$ and for $\lambda \in (0, \min(\alpha, dp))$ with non-integer λ/p .

Although $\|\xi\|^\lambda$ is not necessarily integrable, the integral in the right-hand side of (6.1) is well defined for all $\lambda > 0$. The following result describes the limiting behaviour of the λ -moment of $\|\xi\|$ as $\lambda \uparrow \alpha$.

Corollary 6.8. *If ξ is $S\alpha S$ with $\alpha \in (0, 2)$ and spectral measure σ , then*

$$\lim_{\lambda \uparrow \alpha} \frac{\mathbf{E} \|\xi\|^\lambda}{\Gamma(1 - \frac{\lambda}{\alpha})} = \frac{2^\alpha}{\sqrt{\pi}} \frac{\Gamma(\frac{d+\alpha}{2}) \Gamma(\frac{\alpha+1}{2}) \Gamma(\frac{d}{2})}{\Gamma(1 - \frac{\alpha}{2}) \Gamma(\frac{d-1}{2}) \Gamma(\frac{d+\alpha+1}{2})} \sigma(\mathbb{S}^{d-1}).$$

Proof. It suffices to refer to (6.1) together with

$$\int_{\mathbb{S}^{d-1}} \|u\|_F^\alpha du = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\langle u, y \rangle|^\alpha \sigma(dy) du = \int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{S}^{d-1}} |\langle u, y \rangle|^\alpha du \right) \sigma(dy),$$

and use the fact that

$$\int_{\mathbb{S}^{d-1}} |\langle u, y \rangle|^\alpha du = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} B\left(\frac{\alpha+1}{2}, \frac{d}{2}\right) \|y\|^\alpha,$$

where B is the beta-function. \square

The *intersection body* of a centred star body L is the star body IL such that

$$\|u\|_{IL}^{-1} = \text{Vol}_{d-1}(L \cap u^\perp), \quad u \in \mathbb{S}^{d-1}.$$

For $\xi \in \mathbb{R}^d$, define $\|\xi\|_{IL}^{-1} = \|\xi\| \text{Vol}_{d-1}(L \cap \xi^\perp)$, where ξ^\perp is the $(d-1)$ -dimensional subspace orthogonal to ξ .

Theorem 6.9. *If ξ is $S\alpha S$ with associated star body F and $d \geq 2$, then*

$$\mathbf{E} \|\xi\|_{IF}^{-1} = \frac{1}{\pi(d-1)} \Gamma\left(1 + \frac{1}{\alpha}\right) |F|.$$

Proof. It is known [26, p. 72] that the Fourier transform of $g(x) = \|x\|_L^{-1}$ for a star body L is given by $(2\pi)^d \|u\|_L^{-d+1} / (\pi(d-1))$. Thus,

$$\begin{aligned} \mathbf{E} \|\xi\|_{IL}^{-1} &= \frac{1}{\pi(d-1)} \mathbf{E} \int_{\zeta F} \|x\|_L^{-d+1} dx \\ &= \frac{1}{\pi(d-1)} \Gamma\left(1 + \frac{1}{\alpha}\right) \int_{\mathbb{S}^{d-1}} \rho_F(u) \rho_L(u)^{d-1} du. \end{aligned}$$

If $F = L$, the integral becomes the polar coordinate representation of the volume of F . \square

Similarly to Theorem 6.9 and using [26, Th. 4.6] it is possible to deduce that

$$\mathbf{E} \|\xi\|_{I_k L}^{-k} = \frac{1}{(2\pi)^k (d-k)} \Gamma\left(1 + \frac{k}{\alpha}\right) \int_{\mathbb{S}^{d-1}} \rho_L(u)^{d-k} \rho_F(u)^k du,$$

where $I_k L$ is the k -intersection body of L , so that this moment is proportional to the volume of F if $L = F$. Note that these intersection bodies are defined from $\text{Vol}_k(I_k L \cap H^\perp) = \text{Vol}_{n-k}(L \cap H)$ for each $(n-k)$ -dimensional subspace H , which differs by a factor of 2 from the definition of IL for $k = 1$.

6.2 Mixed moments

The following result deals with joint moments of the coordinates of ξ . For a function $g(x_1, \dots, x_d)$ and $j = 1, \dots, d$ denote

$$\Delta_j g(x) = g(x) - g(x|_j),$$

where $x|_j$ is x with the j th coordinate replaced by zero.

Theorem 6.10. *If ξ is $S\alpha S$ and $\lambda_1, \dots, \lambda_d$ are positive numbers with $\lambda = \sum \lambda_i < \alpha$, then*

$$\begin{aligned} \mathbf{E}(|\xi_1|^{\lambda_1} \dots |\xi_d|^{\lambda_d}) &= 2^{\lambda-d} \frac{(-1)^d}{\pi^{d/2}} \Gamma(1 - \frac{\lambda}{\alpha}) \prod_{i=1}^d \frac{\lambda_i \Gamma(\frac{\lambda_i+1}{2})}{\Gamma(1 - \frac{\lambda_i}{2})} \\ &\times \int_{\mathbb{R}^d} |u_1|^{-\lambda_1-1} \dots |u_d|^{-\lambda_d-1} (\Delta_1 \dots \Delta_d \mathbb{I}_F(u)) du. \end{aligned} \quad (6.5)$$

Proof. The result follows from the formula for the Fourier transform of $|x|^\lambda$ as $-2 \sin(\lambda\pi/2) \Gamma(\lambda+1) |u|^{-\lambda-1}$ (see [14, Sec. II.2.3]) and the fact that the Fourier transform of the direct product $\prod |x_i|^{\lambda_i}$ is the direct product of Fourier transforms, see [14, Sec. II.3.2]. The expression $\Delta_1 \dots \Delta_d \mathbb{I}_F(u)$ appears as a result of the regularisation procedure, see [14, Sec. I.3.2]. Finally, one needs the expression for the $(-\lambda)$ th moment of ζ from (5.2) and the fact that

$$\frac{1}{\pi} \sin \frac{\lambda\pi}{2} \Gamma(\lambda+1) = \frac{\lambda 2^{\lambda-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{\lambda+1}{2})}{\Gamma(1 - \frac{\lambda}{2})}.$$

□

Since $\Delta_1 \dots \Delta_d \mathbb{I}_F(u)$ vanishes in a neighbourhood of the origin, the integral in (6.5) is well defined. If $d = 2$, then (6.5) turns into

$$\begin{aligned} \mathbf{E}(|\xi_1|^{\lambda_1} |\xi_2|^{\lambda_2}) &= \frac{2^{\lambda-2}}{\pi} \Gamma(1 - \frac{\lambda}{\alpha}) \prod_{i=1}^2 \frac{\lambda_i \Gamma(\frac{\lambda_i+1}{2})}{\Gamma(1 - \frac{\lambda_i}{2})} \\ &\times \int_{\mathbb{R}^2} |u_1|^{-\lambda_1-1} |u_2|^{-\lambda_2-1} \left(\mathbb{I}_F(u_1, u_2) - \mathbb{I}_F(0, u_2) - \mathbb{I}_F(u_1, 0) + 1 \right) du_1 du_2. \end{aligned}$$

The *signed power* of a real number t is defined by

$$t^{(\lambda)} = |t|^\lambda \text{sign}(t), \quad (6.6)$$

where $\text{sign}(t)$ is the sign of t .

Theorem 6.11. *If ξ is $S\alpha S$ in \mathbb{R}^d for an even d and $\lambda_1, \dots, \lambda_d$ are non-negative numbers, with none of them being 1 and such that $\lambda = \sum \lambda_i < \alpha$, then*

$$\begin{aligned} \mathbf{E}(\xi_1^{\langle \lambda_1 \rangle} \dots \xi_d^{\langle \lambda_d \rangle}) &= \frac{2^{\lambda} i^d}{\pi^{d/2}} \Gamma(1 - \frac{\lambda}{\alpha}) \prod_{i=1}^d \frac{\Gamma(1 + \frac{\lambda_i}{2})}{\Gamma(\frac{1}{2} - \frac{\lambda_i}{2})} \\ &\quad \times \int_F u_1^{\langle -\lambda_1 - 1 \rangle} \dots u_d^{\langle -\lambda_d - 1 \rangle} du, \quad (6.7) \end{aligned}$$

where the integral is understood as its principal value, i.e. the limit of the integral over $F \setminus \varepsilon B$ as $\varepsilon \rightarrow 0$. The mixed moments vanish if d is odd.

Proof. Use the formula $2i\Gamma(\lambda+1)\cos(\lambda\pi/2)u^{\langle -\lambda-1 \rangle}$ for the Fourier transform of the function $x^{\langle \lambda \rangle}$ with a non-integer λ , see [14, Sec. II.2.3] and identities for the Gamma function. \square

For a centred star body F in \mathbb{R}^k denote

$$\mathcal{I}(F) = \int_F \frac{du}{u_1 \dots u_k},$$

where the integral is understood as its principal value (note that F contains a neighbourhood of the origin in \mathbb{R}^k). Note that $\mathcal{I}(AF) = \mathcal{I}(F)$ for each diagonal matrix A with non-vanishing entries and

$$\mathcal{I}(F) = \int_{\mathbb{S}^{k-1}} \frac{\log \|v\|_F}{v_1 \dots v_k} dv.$$

Corollary 6.12. *If ξ is $S\alpha S$ random vector in \mathbb{R}^d and d is even, then*

$$\mathbf{E} \operatorname{sign}(\xi_1 \dots \xi_d) = \frac{i^d}{\pi^d} \mathcal{I}(F). \quad (6.8)$$

Since the left-hand side of (6.8) does not exceed one in absolute value, we obtain an inequality $|\mathcal{I}(F)| \leq \pi^d$ valid for all centred star bodies $F \subset \mathbb{R}^d$. Note that the expectation in (6.8) does not depend on α . If $d = 2$, then

$$\mathbf{E} \operatorname{sign}(\xi_1 \xi_2) = -\frac{1}{\pi^2} \mathcal{I}(F). \quad (6.9)$$

Note that in (6.7) at most one of the λ_i 's equals 1, since their total sum is strictly less than 2. The case of $\lambda_i = 1$ needs a special treatment, since the Fourier transform of x is given by $(-2\pi i)\delta'(u)$, i.e. it acts as $(-2\pi i)$ times the derivative of the test function at zero.

Theorem 6.13. *Let ξ be $S\alpha S$ with $\alpha \in (1, 2]$, the associated star body F and the associated zonoid $K = F^*$. If d is even and $\lambda_2, \dots, \lambda_d$ are non-negative numbers such that $\lambda = 1 + \lambda_2 + \dots + \lambda_d < \alpha$, then*

$$\mathbf{E}(\xi_1 \xi_2^{\langle \lambda_2 \rangle} \dots \xi_d^{\langle \lambda_d \rangle}) = -\frac{\alpha 2^{\lambda-1} i^d}{\pi^{(d-1)/2}} \Gamma(2 - \frac{\lambda}{\alpha}) \prod_{i=2}^d \frac{\Gamma(1 + \frac{\lambda_i}{2})}{\Gamma(\frac{1}{2} - \frac{\lambda_i}{2})}$$

$$\int_{F \cap e_1^\perp} u_2^{\langle -\lambda_2-1 \rangle} \dots u_d^{\langle -\lambda_d-1 \rangle} \|u\|_F^{\alpha-1} h(T(K, u), e_1) du_2 \dots du_d, \quad (6.10)$$

where $e_1 = (1, 0, \dots, 0)$, $T(K, u)$ is the support set of K in direction u , see (4.3), and the integral is understood as its principal value. The mixed moments vanish if d is odd.

Proof. Since $\alpha > 1$, Theorem 4.7 implies that the support function of K is differentiable. It is well known (see [48, Th. 1.7.2]) that the directional derivative of the support function is given by

$$\lim_{s \downarrow 0} \frac{h(K, u + vs) - h(K, u)}{s} = h(T(K, u), v). \quad (6.11)$$

This formula for $v = e_1$ yields that the Fourier transform \widehat{x}_1 acts on $\varphi(u) = e^{-h(K, u)^\alpha}$ as

$$(-2\pi i) e^{-h(K, u|_1)^\alpha} \alpha \|u|_1\|_F^{\alpha-1} h(T(K, u|_1), e_1),$$

where $u|_1 = (0, u_2, \dots, u_d)$. The remainder of the proof relies on the formulae for Fourier transforms of the signed powers as in Theorem 6.11. \square

The following result gives a formula for the probability that $S\alpha S$ vector ξ takes a value from a polyhedral cone.

Theorem 6.14. *If ξ is $S\alpha S$ with associated star body F , then for each invertible matrix A we have*

$$\mathbf{P}\{\xi \in A\mathbb{R}_+^d\} = \frac{1}{(2\pi)^d} \sum_{m=0}^{\lfloor \frac{d}{2} \rfloor} \pi^{d-2m} (-1)^m \sum_{\{i_1, \dots, i_{2m}\} \subset \{1, \dots, d\}} \mathcal{I}((A^\top F) \cap H_{i_1, \dots, i_{2m}}),$$

where $H_{i_1, \dots, i_{2m}}$ is the hyperplane of dimension $2m$ spanned by the basis vectors $e_{i_1}, \dots, e_{i_{2m}}$.

Proof. By [14, II.2.3 (6)], the Fourier transform of the generalised function $x_{j+}^0 = \mathbb{1}_{x_j \geq 0}$ is given by $iu_j^{-1} + \pi\delta(u_j)$, so that the Fourier transform of $\mathbb{1}_{x \in \mathbb{R}_+^d}$ is the product

$$\prod_{k=1}^d \left(\frac{i}{u_k} + \pi\delta(u_k) \right).$$

Now it suffices to open the parentheses in the product and use the fact that the delta function $\delta(u_k)$ applied to the indicator of F yields 1.

Finally, it remains to note that $\mathbf{P}\{\xi \in A\mathbb{R}_+^d\} = \mathbf{P}\{A^{-1}\xi \in \mathbb{R}_+^d\}$ and that $A^{-1}\xi$ has the associated star body $A^\top F$. \square

It is easy to see that the result of Theorem 6.14 corresponds to (6.9) if $d = 2$. In a similar manner it is possible to compute mixed moments of the positive parts of the components of ξ .

6.3 Integrals of the density

The following result expresses the integrals of the density over 1-dimensional subspaces of \mathbb{R}^d .

Theorem 6.15. *If f is the density of $S\alpha S$ law, then, for each unit vector u ,*

$$\int_{\mathbb{R}} f(tu)dt = \frac{1}{(2\pi)^{d-1}} \Gamma\left(1 + \frac{d-1}{\alpha}\right) A_{F,u}, \quad (6.12)$$

where $A_{F,u} = \text{Vol}_{d-1}(F \cap u^\perp)$ is the $(d-1)$ -dimensional Lebesgue measure of the intersection of F with the subspace orthogonal to u .

Proof. Using the technique of generalised functions, it is possible to calculate the Fourier transform of the function $g = \delta_{\langle u, x \rangle}$ for a fixed unit vector u as $(\hat{g}, \psi) = (g, \hat{\psi})$ for any test function ψ and its Fourier transform $\hat{\psi}$, see [14]. A direct calculation shows that

$$(\hat{g}, \psi) = (2\pi)^{d-1} \int_{\mathbb{R}} \psi(tu)dt.$$

By applying this expression to the density f and using (5.1) we obtain that

$$(g, \varphi) = \mathbf{E} \text{Vol}_{d-1}((\zeta F) \cap u^\perp) = \Gamma\left(1 + \frac{d-1}{\alpha}\right) A_{F,u}.$$

\square

The question, if $A_{F_1,u} \leq A_{F_2,u}$ for convex sets F_1 and F_2 and all $u \in \mathbb{S}^{d-1}$ implies that the volume of F_1 is smaller than the volume of F_2 is known in convex geometry under the name of the *Busemann–Petty problem*. This problem has been recently completely solved (see, e.g. [13] for the solution based on the Fourier analysis) by establishing that the answer is affirmative only in dimensions at most 4. The sets F that appear as associated star bodies of $S\alpha S$ distributions are L_p -balls and so are intersection bodies, for which the Busemann–Petty problem has an affirmative answer in all dimensions, see [26, Sec. 4.3]. In application to stable distributions this means that if two $S\alpha S$ densities f_1 and f_2 with the same characteristic exponent satisfy

$$\int_{\mathbb{R}} f_1(tu)dt \leq \int_{\mathbb{R}} f_2(tu)dt, \quad u \in \mathbb{S}^{d-1},$$

then $f_1(0) \leq f_2(0)$. Recall that by (5.4) the value of the density at the origin is proportional to the volume of F .

It is also possible to consider the intersection of F with a subspace H_k of dimension k and obtain that (see also [26, Lemma 3.24])

$$\int_{H_k^\perp} f(x)dx = \frac{1}{(2\pi)^k} \Gamma(1 + \frac{k}{\alpha}) \text{Vol}_k(F \cap H_k),$$

which yields (5.4) for $k = d$ and (6.12) for $k = d - 1$. For $k = 1$ we get

$$\int_{\langle u, x \rangle = 0} f(x)dx = \frac{1}{\pi} \Gamma(1 + \frac{1}{\alpha}) \rho_F(u) = \frac{1}{\pi} \Gamma(1 + \frac{1}{\alpha}) \|u\|_F^{-1}.$$

It is also possible to express the integral of the type $\int_0^\infty f(tu)t^{d+\lambda-1}dt$ by means of the action of the generalised function $|t|^{-d-\lambda}$ on the test function $A_{F,u}(t) = \text{Vol}_{d-1}(F \cap (u^\perp + tu))$. This yields the $L_{d+\lambda}$ -star of ξ , see [35]. In particular, the L_1 -star of ξ has the radial function (6.12) and so is proportional to the intersection body of F .

6.4 Scalar products and zonoids of random vectors

Moments of scalar products of ξ with unit vector u can be easily calculated using the Fourier transform of the generalised function $|\langle x, u \rangle|^\lambda$, see [26, Lemma 3.14], or by the explicit calculation of the moments of the $S\alpha S$ random variable $\langle \xi, u \rangle$.

Theorem 6.16. *If ξ is $S\alpha S$ and $u \in \mathbb{S}^{d-1}$, then*

$$\mathbf{E} |\langle \xi, u \rangle|^\lambda = 2^\lambda \frac{\Gamma(\frac{\lambda+1}{2})}{\sqrt{\pi}} \frac{\Gamma(1 - \frac{\lambda}{\alpha})}{\Gamma(1 - \frac{\lambda}{2})} \|u\|_F^\lambda$$

for $\lambda \in (-1, \alpha)$.

The *zonoid* of an integrable random vector ξ is defined as the expectation of the random segment $X = [0, \xi]$, see [28, 44]. Representing X as $\frac{1}{2}\xi + [-\frac{1}{2}\xi, \frac{1}{2}\xi]$, the expectation of X can be found from

$$h(\mathbf{E} X, u) = \frac{1}{2} \langle \mathbf{E} \xi, u \rangle + \frac{1}{2} \mathbf{E} |\langle \xi, u \rangle|.$$

If ξ is $S\alpha S$ with $\alpha \in (1, 2]$, then $\mathbf{E} \xi = 0$ and Theorem 6.16 with $\lambda = 1$ yields that

$$h(\mathbf{E} X, u) = \frac{1}{\pi} \Gamma(1 - \frac{1}{\alpha}) \|u\|_F,$$

so that

$$\mathbf{E} X = \frac{1}{\pi} \Gamma(1 - \frac{1}{\alpha}) K.$$

Thus, the zonoid of ξ in the sense of [44] coincides with the rescaled associated zonoid of ξ . The volume of the zonoid $\mathbf{E} X$ is closely related to the expectation of a random determinant whose columns are i.i.d. realisations of ξ . Note also various statistical applications of zonoids of random vectors, e.g. for trimming of multivariate observations, see [27]. Furthermore, the associated zonoid of ξ can be estimated as the rescaled zonoid of ξ , e.g. by evaluating the Minkowski average of $[0, \xi^{(i)}]$ for the i.i.d. sample $\xi^{(1)}, \dots, \xi^{(n)}$. In order to recover the spectral measure from the associated zonoid, one has to use the inversion formula for the p -cosine transform (see [25]) combined with a smoothing operation applied to the support function of K .

7 Stable laws in \mathbb{R}_+^d

7.1 Power sums

Stable laws with all non-negative components (or one-sided laws) are traditionally called totally skewed to the right. However, if one considers them on the semigroup \mathbb{R}_+^d with addition, these distributions can be also called

symmetric stable laws, since this semigroup has the identical involution, see [5]. Still we retain the term $S\alpha S$ only for origin-symmetric stable laws in the whole space. The Laplace transform of one-sided strictly stable law is given by

$$\mathbf{E} e^{-\langle u, \xi \rangle} = \exp \left\{ - \int_{\mathbb{S}_+^{d-1}} \langle u, y \rangle^\alpha \sigma(dy) \right\}, \quad u \in \mathbb{R}_+^d,$$

where $\alpha \in (0, 1)$ and the spectral measure σ on $\mathbb{S}_+^{d-1} = \mathbb{S}^{d-1} \cap \mathbb{R}_+^d$ is unique. It is clearly possible to write

$$\mathbf{E} e^{-\langle u, \xi \rangle} = e^{-\|u\|_F^\alpha} \quad (7.1)$$

for a centred star-shaped (not necessarily convex) set F from (3.2) with the spectral measure obtained by taking all possible reflections of σ with respect to coordinate planes.

Below we show how to develop an alternative representation of the Laplace exponent using convex sets. For this purpose, it is useful to work with generalised power sums. For $p \in (0, \infty)$, the p -sum of two non-negative numbers s and t is defined by

$$s +_p t = (s^p + t^p)^{1/p}.$$

If $p = \infty$, this operation turns into the maximum of s and t . The p -sum $x +_p y$ for $x, y \in \mathbb{R}_+^d$ is defined coordinatewisely as

$$x +_p y = (x_1 +_p y_1, \dots, x_d +_p y_d).$$

Random vector ξ in \mathbb{R}_+^d is *strictly stable for p -sums* with characteristic exponent $\alpha \neq 0$ if

$$a^{1/\alpha} \xi_1 +_p b^{1/\alpha} \xi_2 \stackrel{D}{=} (a + b)^{1/\alpha} \xi \quad (7.2)$$

for all $a, b > 0$ and ξ_1, ξ_2 being independent copies of ξ . The special cases correspond to the usual stability for *arithmetic* sums ($p = 1$) and *max-stability* ($p = \infty$). The general results from [5] concerning stable distributions on abelian semigroups imply that $\alpha \in (0, p]$. It is easy to see that ξ satisfies (7.2) with $p \in (0, \infty)$ if and only if ξ^p is strictly stable for arithmetic sums with the characteristic exponent $\alpha' = \alpha/p$. Note that a power of a vector is always understood coordinatewisely, i.e. $\xi^p = (\xi_1^p, \dots, \xi_d^p)$. The p th signed power of a set $M \subset \mathbb{R}^d$ is defined as

$$M^{(p)} = \{x^{(p)} : x \in M\}, \quad (7.3)$$

where $p > 0$ and $x^{(p)}$ is the vector of signed powers of the components of x , see (6.6).

The analytical tools for p -sums rely on the concept of characters on semigroups, see [3]. A *character* χ is a homomorphism between a semigroup and the unit complex disk with multiplication operation. The involution operation on the semigroup corresponds to the complex conjugation operation on characters. The involution is identical if and only if all characters are real-valued. In particular, in \mathbb{R}_+ with (arithmetic) addition (and identical involution) the characters are $\chi(x) = e^{-tx}$; in \mathbb{R} with addition (so that the involution is the negation) we set $\chi(x) = e^{itx}$; in \mathbb{R}_+ with the coordinatewise maximum (and identical involution) the characters are $\chi(x) = \mathbb{I}_{x \leq t}$ for $t \geq 0$.

If the characters separate all points, i.e. if $x_1 \neq x_2$ implies $\chi(x_1) \neq \chi(x_2)$ for some χ and the characters generate the Borel σ -algebra, then the *Laplace transform* $\chi \mapsto \mathbf{E}\chi(\xi)$ characterises uniquely the distribution of a random element ξ , see [5, Th. 5.3]. In special cases one obtains the characteristic function, the Laplace transform or the cumulative distribution function.

In \mathbb{R}_+^d with the p -sum operation the characters are given by

$$\chi_u(x) = \exp \left\{ - \sum_{i=1}^d (x_i u_i)^p \right\}, \quad x \in \mathbb{R}_+^d, \quad (7.4)$$

for $u \in \mathbb{R}_+^d$ if p is finite. If $p = \infty$, the characters are

$$\chi_u(x) = \begin{cases} 1 & \text{if } x_i u_i \leq 1 \text{ for all } i = 1, \dots, d, \\ 0 & \text{otherwise,} \end{cases} \quad (7.5)$$

for $u \in \mathbb{R}_+^d$, so that $\mathbf{E}\chi_u(\xi) = \mathbf{P}\{\xi \leq u^{-1}\}$ with $u^{-1} = (u_1^{-1}, \dots, u_d^{-1})$.

7.2 $L_1(p)$ -zonoids

Let

$$yM = \{(y_1 x_1, \dots, y_d x_d) : x \in M\} \quad (7.6)$$

denote the set $M \subset \mathbb{R}^d$ rescaled by a vector $y \in \mathbb{R}^d$ and a set $M \subset \mathbb{R}^d$.

Definition 7.1. Let σ be a finite measure on \mathbb{S}_+^{d-1} with total mass c . Define η to be a random vector distributed according to $c^{-1}\sigma$. The set $K = c\mathbf{E}X$ for

$$X = \eta B_q^d = \{(\eta_1 v_1, \dots, \eta_d v_d) : \|v\|_q \leq 1, v \in \mathbb{R}^d\} \quad (7.7)$$

with $p^{-1} + q^{-1} = 1$ for $p \geq 1$ is said to be $L_1(p)$ -zonoid with spectral measure σ .

Definition 7.1 can be reformulated for a probability measure σ on \mathbb{R}^d and the corresponding random vector η . In this case $K = \mathbf{E}(\eta B_q^d)$ is also called the $L_1(p)$ -zonoid generated by η .

Note that X from (7.7) is a rescaled ℓ_q -ball B_q^d . If $p = \infty$, then X becomes a rescaled crosspolytope. More generally, taking the Firey α -expectation $\mathbf{E}_\alpha X$ yields an $L_\alpha(p)$ -zonoid. The conventional and L_p -zonoids are not members of these new families. It is however possible to define a family of sets that includes all zonoids introduced so far.

Definition 7.2. Let M be a centred star-shaped set in \mathbb{R}^d and let η be a random vector in \mathbb{R}^d with $\mathbf{E} \|\eta\|^p < \infty$ for $p \geq 1$. The Firey p -expectation of ηM is called $L_p(M)$ -zonoid.

If M is the segment with end-points $\pm(1, \dots, 1)$, then Definition 7.2 yields the family of L_p -zonoids. The case of M being simplices of varying dimension was considered in [46]. If M is an ℓ_q -ball, we arrive at Definition 7.1. Although Definition 7.2 with a general M may be of geometric interest, we do not pursue its study in this paper.

It is obvious that $L_1(p)$ -zonoids are *plane-symmetric*, i.e. they are symmetric with respect to all coordinate planes. The following proposition shows that the $L_1(p)$ -zonoid is actually determined by the vector $|\eta| = (|\eta_1|, \dots, |\eta_d|)$ of the absolute values of $\eta = (\eta_1, \dots, \eta_d)$. This means that it suffices to consider only spectral measures on \mathbb{R}_+^d .

Proposition 7.3. If $|\eta'|$ and $|\eta''|$ share the same distribution, then the $L_1(p)$ -zonoids generated by η' and η'' coincide.

Proof. It suffices to notice that $\eta' B_q^d$ and $\eta'' B_q^d$ coincide in distribution. \square

Theorem 7.4. A random vector $\xi \in \mathbb{R}_+^d$ is strictly stable for p -sums with $\alpha = 1$, $p \in (1, \infty]$ and spectral measure σ if and only if

$$\mathbf{E} \chi_u(\xi) = e^{-h(K, u)}, \quad u \in \mathbb{R}_+^d, \quad (7.8)$$

for an $L_1(p)$ -zonoid K with spectral measure $\Gamma(1 - \frac{1}{p})\sigma$, where χ_u is given by (7.4) if p is finite and by (7.5) if $p = \infty$.

Proof. Assume that $p \in (1, \infty)$ and consider $\alpha \in [1, p)$. The general results from [5, Sec. 5.3] imply that the Laplace transform of a strictly stable for p -sums random vector in \mathbb{R}_+^d with characteristic exponent α is given by

$$\mathbf{E} \chi_u(\xi) = e^{-\psi(u)},$$

where

$$\psi(u) = \int_{\mathbb{R}_+^d} (1 - \chi_u(x)) \Lambda(dx), \quad u \in \mathbb{R}_+^d, \quad (7.9)$$

and Λ is the Lévy measure ξ . The Lévy measure admits the polar decomposition as $\alpha t^{-\alpha-d} dt \sigma(dy)$ for $x = ty$, so that a change of variables in the integral yields that

$$\psi(u) = \int_{\mathbb{S}_+^{d-1}} \left(\sum_{i=1}^d (u_i y_i)^p \right)^{\alpha/p} \sigma(dy) \frac{\alpha}{p} \int_0^\infty (1 - e^{-s}) s^{-\frac{\alpha}{p}-1} ds.$$

Thus,

$$\psi(u) = \Gamma(1 - \frac{\alpha}{p}) \int_{\mathbb{S}_+^{d-1}} \left(\sum_{i=1}^d (u_i y_i)^p \right)^{\alpha/p} \sigma(dy). \quad (7.10)$$

Since the ℓ_p -norm of u can be written as $(\sum u_i^p)^{1/p} = h(B_q^d, u)$,

$$\left(\sum_{i=1}^d (y_i u_i)^p \right)^{1/p} = h(y B_q^d, u),$$

where $y B_q^d$ is defined as in (7.6). If $\alpha = 1$, then

$$\psi(u) = \Gamma(1 - \frac{\alpha}{p}) \int_{\mathbb{S}_+^{d-1}} h(y B_q^d, u) \sigma(dy),$$

i.e. $\psi(u) = h(c \mathbf{E} X, u)$ for $u \in \mathbb{R}_+^d$, where $c = \sigma(\mathbb{S}_+^{d-1}) \Gamma(1 - \frac{1}{p})$ and X given by (7.7) with $p^{-1} + q^{-1} = 1$ and η distributed according to the normalised σ . The case $p = \infty$ is considered similarly, see [43] for the special study of max-stable laws. \square

Remark 7.5 (Arithmetic sums). Although the conventional case of arithmetic sums ($p = 1$) is not covered by Theorem 7.4, (7.8) also holds. Then $q =$

∞ , so that (7.7) reads $X = \times_{i=1}^d [-\eta_i, \eta_i]$ for $\eta \in \mathbb{R}_+^d$. Thus $K = \mathbf{E} X = \times_{i=1}^d [-\mathbf{E} \eta_i, \mathbf{E} \eta_i]$, so that

$$h(K, u) = \sum u_i \mathbf{E} \eta_i, \quad u \in \mathbb{R}_+^d,$$

i.e. ξ is deterministic. Indeed, one-sided strictly stable laws with $\alpha = 1$ are necessarily degenerated.

Example 7.6 (Max-zonoids). Let ξ be a max-stable random vector in \mathbb{R}_+^d , whose marginals are unit Fréchet, i.e. ξ is stable with respect to the coordinatewise maximum operation and exponent $\alpha = 1$. The Laplace transform of ξ is the cumulative distribution function at the point u^{-1} and

$$\mathbf{P}\{\xi \leq u^{-1}\} = e^{-h(K, u)}$$

for an $L_1(\infty)$ -zonoid K , i.e. K is the expectation of the randomly rescaled crosspolytope in \mathbb{R}^d . These zonoids (more exactly their intersections with \mathbb{R}_+^d) have been explored in [43] in view of the studies of max-stable distributions, and so are called there *max-zonoids*.

Example 7.7 ($p = 2$). If ξ is strictly stable for p -sums with $p = 2$ and $\alpha = 1$, then the Laplace transform of ξ is given by

$$\mathbf{E} \exp \left\{ - \sum (\xi_i u_i)^2 \right\} = e^{-h(K, u)}.$$

The $L_1(2)$ -zonoid K is the selection expectation of a randomly rescaled Euclidean ball, i.e. the centred plane-symmetric random ellipsoid.

Example 7.8. In dimension $d = 2$ it is possible to calculate the support function of X from (7.7) for $u = (u_1, u_2) \in \mathbb{R}_+^2$ as

$$h(X, u) = \sup \{ u_1 \eta_1 \cos^{2/q} \theta + u_2 \eta_2 \sin^{2/q} \theta : 0 \leq \theta \leq \frac{\pi}{2} \}.$$

Substituting the value of the critical point $\tan \theta = (u_2 \eta_2 / u_1 \eta_1)^{p/2}$ and noticing that $\eta_1, \eta_2 \geq 0$ we arrive at

$$h(K, u) = \mathbf{E} h(X, u) = \mathbf{E} \frac{(u_1 \eta_1)^p + (u_2 \eta_2)^p}{(u_1 \eta_1 + u_2 \eta_2)^{p-1}}.$$

Example 7.9 (Completely dependent and independent cases). If η is deterministic, then the corresponding $L_1(p)$ -zonoid is a rescaled ℓ_q -ball and the coordinates of ξ are completely dependent.

A centred parallelepiped $\times_{i=1}^d [-a_i, a_i]$ is an $L_1(p)$ -zonoid for each $p \geq 1$. To check this, it suffices to take the spectral measure concentrated at the unit basis vectors e_1, \dots, e_d with masses a_1, \dots, a_d , so that ξ has independent components. For instance, X from (7.7) becomes the segment $[-e_i, e_i]$ if $\eta = e_i$. Therefore, polytopes may be $L_1(p)$ -zonoids for $p > 1$, cf Theorem 4.7.

Thus, η equal to one of the basic vectors results in X from (7.7) being a segment, while any η from the interior of \mathbb{R}_+^d results in X being a rescaled ℓ_q -ball. By combining such values of η it is easy to construct further examples of $L_1(p)$ -zonoids. For instance, if η takes the values $(2, \dots, 2)$ and $(2, 0, \dots, 0)$ with equal probabilities $1/2$, then the corresponding $L_1(p)$ -zonoid is the Minkowski sum of the unit ℓ_q -ball and the segment with end-points $\pm(1, 0, \dots, 0)$.

7.3 One-sided strictly stable laws

The construction based on p -sums makes it possible to provide a geometric interpretation of strictly stable laws for arithmetic sums on \mathbb{R}_+^d and $\alpha \in (0, 1]$.

Theorem 7.10. *A random vector $\xi \in \mathbb{R}_+^d$ is strictly stable (for arithmetic sums) with $\alpha \in (0, 1]$ if and only if the Laplace transform of ξ is given by*

$$\mathbf{E} e^{-\langle \xi, u \rangle} = e^{-h(K, u^\alpha)}, \quad u \in \mathbb{R}_+^d, \quad (7.11)$$

where $u^\alpha = (u_1^\alpha, \dots, u_d^\alpha)$ and K is $L_1(\alpha^{-1})$ -zonoid called the associated zonoid of ξ . The spectral measure of K is $\Gamma(1-\alpha)\sigma$, where σ is the spectral measure of ξ .

Proof. The random vector ξ^α is strictly stable for p -sums with $p = \frac{1}{\alpha}$, so that (7.11) follows from Theorem 7.4. If $\alpha = 1$, the law of ξ is degenerated, see Remark 7.5. \square

In particular, if $\alpha = \frac{1}{2}$, then K from (7.11) is the expectation of a random ellipsoid as in Example 7.7. By comparing (7.11) with (7.1) we see that $\|u^\alpha\|_{F^*} = \|u\|_F^\alpha$, so that $K^* = F^{(\alpha)}$.

Example 7.11 (One-sided sub-stable laws). Let ξ be one-sided stable law in \mathbb{R}_+^d with $\alpha \in (0, 1)$ and let ζ be a non-negative stable random variable with characteristic exponent $\beta \in (0, 1)$. Then $\xi' = \zeta^{1/\alpha} \xi$ has a sub-stable distribution, see Example 4.6. It is easily seen that

$$\mathbf{E} e^{-\langle \xi', u \rangle} = e^{-h(K, u^\alpha)^\beta} = e^{-h(L, u^{\alpha\beta})},$$

where K and L are the associated zonoids of ξ and ξ' respectively, i.e. K is an $L_1(\alpha^{-1})$ -zonoid and L is an $L_1((\alpha\beta)^{-1})$ -zonoid. Note that $\|u\|_F^\beta = \|u^\beta\|_{F^{(\beta)}}$ for any star body F , where $F^{(\beta)}$ is defined by (7.3). Hence $h(K, u^\alpha)^\beta = h(L, u^{\alpha\beta})$ where $L^* = (K^*)^{(\beta)}$.

Theorem 7.12. *If K is an $L_1(p)$ -zonoid for $p \geq 1$, then K is $L_1(r)$ -zonoid for all $r > p$.*

Proof. We refer to the construction from Example 7.11. Assume that K is a parallelepiped, i.e. the components of ξ are independent. Then L^* is the β -power of the crosspolytope K^* . Since the crosspolytope K^* is the (possibly rescaled) ℓ_1 -ball, its β -power $(K^*)^{(\beta)}$ is the (possibly rescaled) $\ell_{1/\beta}$ -ball. Its polar $L = ((K^*)^{(\beta)})^*$ is a (possibly rescaled) $\ell_{1/(1-\beta)}$ -ball. By the construction of Example 7.11, the $\ell_{1/(1-\beta)}$ -ball L is an $L_1((\alpha\beta)^{-1})$ -zonoid for all $\alpha \in (0, 1)$. By setting $q = 1/(1-\beta)$, it is easy to see that ℓ_q -ball is $L_1(r)$ -zonoid for all $r > p$, where $p^{-1} + q^{-1} = 1$.

Thus, ℓ_q -ball can be represented as the expectation of rescaled $\ell_{r'}$ -balls for each $r' < q$. Since each $L_1(p)$ -zonoid is the expectation of the rescaled ℓ_q -ball, it can also be expressed as the expectation of rescaled $\ell_{r'}$ -ball, where r' is associated with r , so that it is also an $L_1(r)$ -zonoid. \square

Thus, the family of $L_1(p)$ -zonoids becomes richer if p increases. The richest one is the family of $L_1(\infty)$ -zonoids (or max-zonoids). In the planar case it includes all plane-symmetric convex sets [43], while in the spaces of higher dimensions this is no longer the case.

The following generalisation of Theorem 7.4 treats general strictly stable laws for p -sums.

Theorem 7.13. *A random vector $\xi \in \mathbb{R}_+^d$ is strictly stable for p -sums with $p \in (0, \infty]$ and the characteristic exponent $\alpha \leq p$ if and only if*

$$\mathbf{E} \chi_u(\xi) = \exp\{-h(K, u^\alpha)\}$$

for an $L_1(p/\alpha)$ -zonoid K .

Proof. If ξ is strictly stable for p -sums with $\alpha \leq p$ and finite p , then ξ^p is $S\alpha'S$ for arithmetic sums with $\alpha' = \alpha/p \in (0, 1]$, so that Theorem 7.10 yields that

$$\mathbf{E} \exp\left\{-\sum (\xi_i u_i)^p\right\} = \exp\{-h(K, u^{\alpha'})^p\} = \exp\{-h(K, u^\alpha)\},$$

where K is an $L_1(p/\alpha)$ -zonoid. The case of $p = \infty$ follows from the fact that $\psi(u)$ from (7.9) is written as

$$\psi(u) = \int_{\mathbb{S}_+^{d-1}} (\max_{1 \leq i \leq d} (u_i y_i))^\alpha \sigma(dy) = \int_{\mathbb{S}_+^{d-1}} \max_{1 \leq i \leq d} (u_i y_i^\alpha) \sigma'(dy)$$

for another measure σ' . □

7.4 Moments of one-sided stable laws

Similar to the Fourier analysis technique in Section 6, it is possible to use the expression for the Laplace transform of one-sided stable law in order to obtain more information about its probability density function f and moments. By integrating the both parts of

$$\int_{\mathbb{R}_+^d} e^{-\langle x, u \rangle} f(x) dx = e^{-h(K, u^\alpha)}, \quad u \in \mathbb{R}_+^d, \quad (7.12)$$

with respect to u with a certain weight, we arrive at various expressions for the moments of ξ . To start with, integrate the both sides of (7.12) over the ray $\{tu : t \geq 0\}$ with weight t^λ for a fixed unit vector u and $\lambda > -1$. Since

$$\int_0^\infty e^{-\langle x, u \rangle t} t^\lambda dt = \langle x, u \rangle^{-\lambda-1} \Gamma(1 + \lambda),$$

the integration of the right-hand side yields that

$$\mathbf{E} \langle \xi, u \rangle^{-\lambda-1} = \frac{1}{\alpha} \frac{\Gamma(\frac{1+\lambda}{\alpha})}{\Gamma(1+\lambda)} h(K, u^\alpha)^{-(\lambda+1)/\alpha}.$$

For instance, if K is the parallelepiped $\times_{i=1}^d [-a_i, a_i]$ (which corresponds to the independent coordinates of ξ), then for $\lambda = 0$ we have

$$\mathbf{E} \langle \xi, u \rangle^{-1} = \Gamma(1 + \frac{1}{\alpha}) \left(\sum a_i u_i^\alpha \right)^{-1/\alpha} = \left(\sum \left(\frac{u_i}{\mathbf{E}(\xi_i^{-1})} \right)^\alpha \right)^{-1/\alpha}.$$

Note that we have used the fact that $\mathbf{E} \xi_i^{-1} = \Gamma(1 + 1/\alpha) a_i^{-1/\alpha}$.

If one performs a similar integration with $\lambda \in (-1 - \alpha, -1)$, it is possible to regularise the integral of $e^{-\langle x, ut \rangle}$ by subtracting the value of the function for $t = 0$ as

$$\int_{\mathbb{R}_+^d} (1 - e^{-\langle x, u \rangle}) f(x) dx = 1 - e^{-h(K, u^\alpha)}.$$

Then, for $\beta \in (0, \alpha)$

$$\mathbf{E}\langle \xi, u \rangle^\beta = \frac{\Gamma(1 - \frac{\beta}{\alpha})}{\Gamma(1 - \beta)} h(K, u^\alpha)^{\beta/\alpha}.$$

Clearly, the above expressions for the moments can be obtained by calculating the moments of one-sided stable random variable $\langle \xi, u \rangle$. Furthermore, similar results can be obtained for random vectors which are strictly stable for p -sums. The case of $p = \infty$ is considered in [43].

The multivariate *Laplace ordering* is introduced in [49] by pointwise ordering of the Laplace transform. Thus, two one-sided strictly stable random vectors with the same characteristic exponent are Laplace ordered if and only if the corresponding associated zonoids are ordered by inclusion. Applications of this ordering for actuarial quantities have been considered in [7].

8 Geometric interpretations of the spectral measure

8.1 p -surface area measures and spectral bodies

Assume that the $S\alpha S$ distribution is *full-dimensional*, i.e. the spectral measure σ is not concentrated on a great sub-sphere of \mathbb{S}^{d-1} . The *Minkowski existence problem* [48, Sec. 7.1] establishes that for each finite positive full-dimensional even Borel measure on the unit sphere there exists a unique centred convex body Q such that $S(Q, \cdot) = \sigma(\cdot)$. Here $S(Q, \cdot)$ is the *surface area measure* of Q which is the unique measure on the unit sphere that satisfies

$$dV_1(Q, L) = \lim_{\varepsilon \downarrow 0} \frac{|Q + \varepsilon L| - |Q|}{\varepsilon} = \int_{\mathbb{S}^{d-1}} h(L, u) S(Q, du),$$

where $V_1(Q, L)$ is called the *mixed volume* of the convex sets Q and L . We refer to [48] for a detailed presentation of the relevant concepts from convex geometry.

The L_p generalisation of the above concepts has been studied in [32, 33]. The p -mixed volume $V_p(Q, L)$ of two convex bodies containing the origin is

defined by

$$\frac{d}{p}V_p(Q, L) = \lim_{\varepsilon \downarrow 0} \frac{|Q +_p \varepsilon^{1/p}L| - |Q|}{\varepsilon} = \frac{1}{p} \int_{\mathbb{S}^{d-1}} h(L, u)^p S_p(Q, du), \quad (8.1)$$

where the p -surface area measure $S_p(Q, \cdot)$ satisfies

$$S_p(Q, du) = h(Q, u)^{1-p} S(Q, du).$$

The p -Minkowski problem is solved in [32, Th. 3.3] by showing that if σ is an even positive Borel measure which is not concentrated on a great subsphere of \mathbb{S}^{d-1} and $p > 1$, $p \neq d$, then there exists a unique centred convex body Q , such that $S_p(Q, \cdot) = \sigma(\cdot)$. By combining this representation with the classical Minkowski problem for $p = 1$, any full-dimensional spectral measure σ corresponding to $S\alpha S$ law ξ with $\alpha \in [1, 2)$ ($\alpha \in [1, 2]$ in dimension $d \geq 3$) can be interpreted as the α -surface area measure of a centred convex body Q , i.e. $\sigma(\cdot) = S_\alpha(Q, \cdot)$. We call Q the *spectral body* of ξ . By (3.2), the Minkowski functional of the associated star body F (or the support function of the associated zonoid K) can be expressed as

$$\|u\|_F^\alpha = h(K, u)^\alpha = \int_{\mathbb{S}^{d-1}} |\langle u, v \rangle|^\alpha S_\alpha(Q, dv).$$

The p th projection body $\Pi_p Q$ of Q is defined in [34] by

$$h(\Pi_p Q, u)^p = \frac{1}{d\kappa_d c_{d-2,p}} \int_{\mathbb{S}^{d-1}} |\langle x, u \rangle|^p S_p(Q, dx),$$

where

$$c_{d,p} = \frac{\kappa_{d+p}}{\kappa_2 \kappa_d \kappa_{p-1}}.$$

The set $\Pi_p Q$ (or its dilated version) is sometimes denoted by $\Gamma_{-p}^* Q$ and is called the polar centroid body, see, e.g. [37, (4.2)]. The normalising constant guarantees that $\Pi_p B = B$ for the unit Euclidean ball. Thus, the associated zonoid K satisfies $K = (d\kappa_d c_{d-2,\alpha})^{1/\alpha} \Pi_p Q$.

The L_p -analogue of the *Petty projection inequality* proved in [34, Th. 2] establishes that

$$|Q|^{(d-p)/p} \cdot |\Pi_p^* Q| \leq \kappa_d^{d/p}.$$

Using the fact that the polar set $\Pi_\alpha^* Q$ is a dilate of the associated star body F of ξ and setting $p = \alpha$, we arrive at the following inequality

$$|Q|^{-1+d/\alpha} \cdot |F| \leq (d c_{d-2,\alpha})^{-d/\alpha} \quad (8.2)$$

valid for $\alpha \in [1, 2]$ with the equality attained in the sub-Gaussian case. Recall that $|F|$ determines the value of the density of the stable law at the origin and provides a bound for the moments of $\|\xi\|$. In the Gaussian case $\alpha = 2$ for $d \geq 3$ and (8.2) reads

$$|Q|^{-1+d/2} \cdot |F| \leq \left(1 + \frac{d}{2}\right)^{-d/2}.$$

8.2 Spectral star body

The following interpretation of the spectral measure is useful for $S\alpha S$ laws with arbitrary $\alpha \in (0, 2]$. Assume that the spectral measure σ of ξ has a positive continuous density on \mathbb{S}^{d-1} with respect to the $(d-1)$ -dimensional surface area measure, so that

$$\sigma(du) = \frac{1}{d+\alpha} \rho_L(u)^{d+\alpha} du$$

for a star body L , which we call the *spectral star body* of ξ . By passing to polar coordinates it is easily seen that

$$\|u\|_F^\alpha = \int_{\mathbb{S}^{d-1}} |\langle u, y \rangle|^\alpha \sigma(dy) = \int_L |\langle u, y \rangle|^\alpha dy.$$

The integral in the right-hand side is related to the p -centroid body $\Gamma_p L$ and its polar $\Gamma_p^* L$ defined (up to a possibly different normalisation) by

$$h(\Gamma_p L, u)^p = \|u\|_{\Gamma_p^* L}^p = \frac{1}{c_{d,p}|L|} \int_L |\langle u, y \rangle|^p dy,$$

see [12, 34] and [35, (6.1)]. Note that $\Gamma_p L$ is convex for all $p \geq 1$. Thus, the associated star body of $S\alpha S$ vector ξ is related to its spectral star body by

$$F = (c_{d,\alpha}|L|)^{-1/\alpha} \Gamma_\alpha^* L.$$

It is proved in [34] that $|\Gamma_p L| \geq |L|$ if $p \geq 1$, which implies the Blaschke-Santaló inequality $|L| \cdot |\Gamma_p^* L| \leq \kappa_d^2$ with equality if and only if L is a centred ellipsoid, see also [38]. If $p = \alpha \geq 1$, then

$$|F| \cdot |L|^{1+d/\alpha} \leq \frac{\kappa_d^2}{c_{d,\alpha}}.$$

The same spectral star body L can be used to construct $S\alpha S$ random vectors $\xi_{\alpha,L}$ with varying characteristic exponent α . If L is convex, then [15, Prop. 2.1.1] yields that there exists a universal constant $c > 0$ such that for all $u \in \mathbb{R}^d$ and $p > 1$

$$\left(\int_{L_1} |\langle u, y \rangle|^p dy \right)^{1/p} \leq cp \int_{L_1} |\langle u, y \rangle| dy,$$

where $L_1 = |L|^{-1/d}L$ has volume 1. Thus

$$\|u\|_{F_{\alpha,L}} \leq c\alpha |L|^{1-1/\alpha} \|u\|_{F_{1,L}},$$

where $F_{\alpha,L}$ is the associated star body of $\xi_{\alpha,L}$. By Theorem 6.1, this yields an inequality between the λ -moments of the norm with $\lambda \in (0, 1)$ for an $S\alpha S$ random vector with $\alpha \geq 1$ and the Cauchy random vector with the same spectral star body. For the Cauchy distribution we have $\alpha = 1$, where a number of further inequalities for the volumes of projection and polar bodies are available, see [12, 48].

If F is the associated star body of an $S\alpha S$ law with convex spectral body L , then [15, Lemma 3.1.1] implies that

$$\|u\|_F^\alpha \geq |L|^{1+\lambda/d} \frac{\Gamma(\alpha+1)\Gamma(d)}{2e\Gamma(\alpha+d+1)} h(L, u)^\alpha, \quad u \in \mathbb{S}^{d-1}.$$

For instance, this inequality may be used in order to obtain lower bounds for the moments of $\|\xi\|$. It also means that the associated zonoid of ξ contains a dilate of L .

It is likely that other geometric properties, e.g. curvature, surface area, other intrinsic volumes, of the spectral sets and associated star bodies have a bearing in view of the studies of $S\alpha S$ laws.

8.3 Spectral sets for one-sided laws

Let ξ be a one-sided strictly stable random vector. Although its spectral measure σ is supported by \mathbb{S}_+^{d-1} , consider its extension on the whole \mathbb{S}^{d-1} in a plane-symmetric way. Note that this extension is full-dimensional, so that the Minkowski existence problem guarantees that $\sigma(\cdot) = S(Q, \cdot)$ for a convex plane-symmetric body Q , also called the spectral body of ξ .

Let M be a centred convex set in \mathbb{R}^d . If η is a random vector distributed according to the normalised $S(Q, \cdot)$ having the total mass c , then the expectation of $c\eta M$ is given from

$$\begin{aligned} h(c \mathbf{E} \eta M, u) &= \int_{\mathbb{S}^{d-1}} h(yM, u) S(Q, dy) \\ &= \int_{\mathbb{S}^{d-1}} h(uM, y) S(Q, dy) = dV_1(Q, uM). \end{aligned}$$

If ξ is one-sided strictly stable random vector with characteristic exponent α and spectral measure σ , then choose M to be the $\ell_{1/(1-\alpha)}$ -ball, so that

$$\mathbf{E} e^{-\langle \xi, u \rangle} = \exp\{-\Gamma(1-\alpha) dV_1(Q, uB_{1/(1-\alpha)})\}.$$

The first Minkowski inequality $V_1(K, L) \geq |K|^{(d-1)/d} |L|^{1/d}$ (see [48, Th. 6.2.1]) together with the formula for the volume of the unit ℓ_p -ball (see [45, p. 11]) imply that

$$\mathbf{E} e^{-\langle \xi, u \rangle} \leq \exp\left\{-\frac{\Gamma(1-\alpha)^2}{\Gamma(d(1-\alpha))^{1/d}} |Q|^{(d-1)/d} u_1 \cdots u_d\right\}.$$

9 Covariation and regression

9.1 Bivariate case

The covariation replaces the concept of covariance for $S\alpha S$ vectors. If $\xi = (\xi_1, \xi_2)$ is $S\alpha S$ in \mathbb{R}^2 with $\alpha > 1$ and the spectral measure σ , then the *covariation* of ξ_1 on ξ_2 is defined by

$$[\xi_1, \xi_2]_\alpha = \int_{\mathbb{S}^1} s_1 s_2^{\langle \alpha-1 \rangle} \sigma(ds),$$

see [47, Sec. 2.7]. It is mentioned in [47, Sec. 2.7] that the covariation can be equivalently defined as

$$[\xi_1, \xi_2]_\alpha = \frac{1}{\alpha} \left. \frac{\partial \sigma^\alpha(t_1, t_2)}{\partial t_1} \right|_{t_1=0, t_2=1}, \quad (9.1)$$

where $\sigma(t_1, t_2)$ is the scale parameter of $Y = t_1 \xi_1 + t_2 \xi_2$, i.e.

$$\sigma^\alpha(t_1, t_2) = \int_{\mathbb{S}^1} |t_1 s_1 + t_2 s_2|^\alpha \sigma(ds). \quad (9.2)$$

Theorem 9.1. *If $\xi = (\xi_1, \xi_2)$ is $S\alpha S$ with $\alpha \in (1, 2]$ and the associated zonoid K , then*

$$[\xi_1, \xi_2]_\alpha = x_1 x_2^{\alpha-1}, \quad (9.3)$$

where $T(K, (0, 1)) = \{(x_1, x_2)\}$ is the support point of K in direction $(0, 1)$, see (4.3).

Proof. By Theorem 4.7, L_p -zonoids with $p > 1$ are strictly convex, so that the support set $T(K, u)$ is indeed a singleton for each direction u . The right-hand side of (9.2) can be identified as $h(K, u)^\alpha$ for K being the associated zonoid of ξ . The partial derivative in the right-hand side of (9.1) then becomes the directional derivative of $h(K, u)$ in direction $(1, 0)$. By [48, Th. 1.7.2] this derivative can be expressed as $h(T(K, (0, 1)), (1, 0))$. Hence

$$[\xi_1, \xi_2]_\alpha = h(K, (0, 1))^{\alpha-1} h(T(K, (0, 1)), (1, 0)) = x_1 x_2^{\alpha-1}.$$

□

It is shown in [47, Lemma 2.7.16] that, for all $p \in (1, \alpha)$,

$$\frac{\mathbf{E}(\xi_1 \xi_2^{\langle p-1 \rangle})}{\mathbf{E} |\xi_2|^p} = \frac{[\xi_1, \xi_2]_\alpha}{[\xi_2, \xi_2]_\alpha}. \quad (9.4)$$

Using Theorems 6.10 and 6.13 it is possible to calculate the moments in the left-hand side explicitly as

$$\begin{aligned} \mathbf{E}(\xi_1 \xi_2^{\langle p-1 \rangle}) &= \frac{\alpha 2^{p-1}}{\sqrt{\pi}} \Gamma(2 - \frac{p}{\alpha}) \frac{\Gamma(1 + \frac{p-1}{\alpha})}{\Gamma(\frac{1}{2} - \frac{p-1}{2})} \frac{2x_1 x_2^{p-1}}{\alpha - p}, \\ \mathbf{E} |\xi_2|^p &= 2^p x_2^p \frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}} \frac{\Gamma(1 - \frac{p}{\alpha})}{\Gamma(1 - \frac{p}{2})}, \end{aligned}$$

where x_1 and x_2 are the coordinates of $T(K, (0, 1))$. By dividing these expressions we arrive at x_1/x_2 , which is exactly the right-hand side of (9.4).

9.2 Multivariate case

The following result provides covariations for random variables that belong to a linear span of an $S\alpha S$ random vector.

Theorem 9.2. *Let ξ be $S\alpha S$ in \mathbb{R}^d with $\alpha \in (1, 2]$. If u', u'' are non-zero vectors in \mathbb{R}^d , then*

$$[\langle \xi, u' \rangle, \langle \xi, u'' \rangle]_\alpha = h(K, u'')^{\alpha-1} h(T(K, u''), u'). \quad (9.5)$$

Proof. The scale parameter of $t_1 \langle \xi, u' \rangle + t_2 \langle \xi, u'' \rangle$ is $h(K, t_1 u' + t_2 u'')$. By differentiating its power with respect to t_1 as in (9.1), we arrive at (9.5). \square

Theorem 9.2 provides an alternative reformulation of [47, Lemma 2.7.5]. The right-hand side of (9.5) considered a function of u' is the support function of the singleton $h(K, u'')^{\alpha-1} T(K, u'')$, and so is additive with respect to u' . In particular, if $u' = (1, 1, 0)$ and $u'' = (0, 0, 1)$ in \mathbb{R}^3 , it yields the additivity of the covariation of $S\alpha S$ random variables with respect to its first argument. Similarly, one deduces the additivity of the covariation with respect to the sum of independent second arguments. Furthermore, the covariations in the left-hand side of (9.5) for all u', u'' determine uniquely the associated zonoid K .

Example 9.3 (ℓ_p -balls). Assume that K is the unit ℓ_α -ball, which corresponds to $S\alpha S$ vector ξ with i.i.d. components. The support point $T(K, u)$ equals the gradient of $\|u\|_\alpha = h(K, u)$, see [48, Cor. 1.7.3]. Therefore $T(K, u) = \{\|u\|_\alpha^{1-\alpha} u^{\langle \alpha-1 \rangle}\}$. By (9.5),

$$[\langle \xi, u' \rangle, \langle \xi, u'' \rangle]_\alpha = \langle u', (u'')^{\langle \alpha-1 \rangle} \rangle.$$

9.3 Regression coefficients and linearity conditions

The covariation is used to build regression models for $S\alpha S$ distributions. By [47, Th. 4.1.2],

$$\mathbf{E}(\xi_1 | \xi_2) = \frac{[\xi_1, \xi_2]_\alpha}{[\xi_2, \xi_2]_\alpha} \xi_2 \quad \text{a.s.}$$

Since $[\xi_2, \xi_2]_\alpha = x_2^\alpha$ for $T(K, (0, 1)) = \{(x_1, x_2)\}$, we obtain

$$\mathbf{E}(\xi_1 | \xi_2) = \frac{x_1}{x_2} \xi_2 \quad \text{a.s.}$$

Thus, the regression line is the line passing through the origin and the support point $T(K, (0, 1))$. Therefore, the regression lines are identical for any two $S\alpha S$ laws that share the same associated zonoid, e.g. for the Gaussian law and its sub-Gaussian variant.

It is known [47, Sec. 4.1] that multiple regression is not always linear for $\alpha \in (1, 2)$. The necessary and sufficient conditions for the linearity given in [40] can be reformulated geometrically as follows.

Consider a convex set K and the one-dimensional subspace H_x spanned by $x \in \mathbb{R}^d$. The *shadow boundary* of K in direction x is the set $\partial(K + H_x) \cap \partial K$, where ∂ denotes the topological boundary in \mathbb{R}^d , see [50, Def. 3.4.7].

Theorem 9.4. *Let (ξ_1, \dots, ξ_d) be an $S\alpha S$ random vector with $\alpha \in (1, 2]$ and the associated zonoid K . Then $\mathbf{E}(\xi_1 | \xi_2, \dots, \xi_d)$ is linear in ξ_2, \dots, ξ_d if and only if the shadow boundary of K in direction $e_1 = (1, 0, \dots, 0)$ is a subset of a $(d - 1)$ -dimensional hyperplane, which does not contain e_1 .*

Proof. By Theorem 3.1 from [40], the conditional expectation is linear if and only if, for all u_2, \dots, u_d ,

$$\left. \frac{\partial}{\partial u_1} \varphi_\xi(u_1, u_2, \dots, u_d) \right|_{u_1=0} = \sum_{i=2}^d a_i \frac{\partial}{\partial u_i} \varphi_\xi(0, u_2, \dots, u_d).$$

By differentiating (4.2) and using [48, Th. 1.7.2] for the directional derivative of the support function, it is easily seen that this holds if and only if

$$h(T(K, u|_1), e_1) = \sum_{i=2}^d a_i h(T(K, u|_1), e_i), \quad (9.6)$$

where $u|_1$ is u with the first coordinate replaced by zero. Since $h(T(K, u|_1), e_i)$ is the i th coordinate of the singleton $T(K, u|_1)$, (9.6) means that $T(K, u|_1)$ is orthogonal to $a = (1, -a_2, \dots, -a_d)$ for all $u|_1 = (0, u_2, \dots, u_d)$. In other words, the shadow boundary of K in direction e_1 lies in the hyperplane orthogonal to a . By the condition, the first coordinate of a is not zero, so that this hyperplane does not contain e_1 . \square

Example 9.5 (Sub-Gaussian laws). If ξ is sub-Gaussian with the norm $\|u\|_E = \langle Cu, u \rangle$, then $K = E^*$ is an ellipsoid. It is easy to see (see [21]) that $T(K, u)$ is the point $C^{-1}u / \sqrt{\langle C^{-1}u, u \rangle}$. Then the condition of Theorem 9.4 holds with $a = Ce_1$.

Corollary 9.6. *Let (ξ_1, \dots, ξ_d) be an $S\alpha S$ random vector with $\alpha \in (1, 2]$, the associated zonoid K and the spectral measure σ . Then $\mathbf{E}(\xi_1 | \xi_2, \dots, \xi_d)$ is linear in ξ_2, \dots, ξ_d if and only if there exists $a \in \mathbb{R}^d$ with non-vanishing first*

coordinate such that one of the following equivalent conditions holds for all u orthogonal to e_1 :

$$\langle \text{grad } h(K, u), a \rangle = 0, \quad (9.7)$$

$$\int_{\mathbb{S}^{d-1}} \langle y, a \rangle \langle y, u \rangle^{\langle \alpha-1 \rangle} \sigma(dy) = 0, \quad (9.8)$$

$$[\langle \xi, a \rangle, \langle \xi, u \rangle]_\alpha = 0. \quad (9.9)$$

Proof. Since the support function of K is differentiable by Theorem 4.7, the support point of K in direction u is given by the gradient of $h(K, u)$, see [48, Cor. 1.7.3]. This yields, (9.7). By differentiating (4.1), it is easy to see that

$$\text{grad } h(K, u) = h(K, u)^{\alpha-1} \int_{\mathbb{S}^{d-1}} y \langle u, u \rangle^{\langle \alpha-1 \rangle} \sigma(dy),$$

so that (9.7) is indeed equivalent to (9.8). Finally, (9.5) implies that

$$[\langle \xi, a \rangle, \langle \xi, u \rangle]_\alpha = \int_{\mathbb{S}^{d-1}} \langle y, a \rangle \langle y, u \rangle^{\langle \alpha-1 \rangle} \sigma(dy).$$

□

The vector ξ has the *multiple regression property* if, for each linear transformation A , the multiple regression of the first coordinate of $\eta = A\xi$ onto the remaining coordinates is linear. Note that η has the associated zonoid AK . By Theorem 9.4, this happens if and only if the shadow boundary of K in each direction is contained in a $(d-1)$ -dimensional hyperplane. W. Blaschke proved in 1916 that for dimension $d \geq 3$ this is the case if and only if K is an ellipsoid, see [50, Th. 3.4.8]. By Theorem 9.4, this geometric result translates into the multiple regression criterion from [47, Prop. 4.1.7].

10 Operations with associated sets

If ξ' and ξ'' are independent $S\alpha S$ with associated star bodies F_1 and F_2 , then $\xi = \xi' + \xi''$ has the characteristic function

$$\mathbf{E} e^{i\langle u, \xi \rangle} = \exp\{-(\|u\|_{F_1}^\alpha + \|u\|_{F_2}^\alpha)\}.$$

Thus, ξ has the associated star body F being the radial sum (also called α -star sum) of F_1 and F_2 , i.e. $F = F_1 \tilde{+}_\alpha F_2$.

Theorem 10.1. *If ξ' and ξ'' are independent $S\alpha S$ in \mathbb{R}^d with $\alpha \in [1, 2]$ and probability densities $f_{\xi'}$ and $f_{\xi''}$ respectively, then the probability density of $\xi = \xi' + \xi''$ satisfies*

$$f_{\xi}(0)^{-\alpha/d} \geq f_{\xi'}(0)^{-\alpha/d} + f_{\xi''}(0)^{-\alpha/d}$$

with the equality if and only if the associated star bodies of ξ' and ξ'' are dilates.

Proof. The result follows from (5.4) and the dual Brunn–Minkowski inequality for radial sums of star bodies, see [33, Prop. 1.12]. \square

The following result concerns approximation by sub-Gaussian laws.

Theorem 10.2. *A law is $S\alpha S$ with $\alpha \in [1, 2]$ if and only if it can be obtained as a weak limit for the sums of independent sub-Gaussian laws with the same characteristic exponent.*

Proof. By [18, Cor. 6.14], each centred convex body F is an L_p -ball with $p \geq 1$ if and only if $\|u\|_F^p$ can be uniformly approximated for u from the unit sphere by finite sums of the form $\|u\|_{E_1}^p + \dots + \|u\|_{E_m}^p$, where E_1, \dots, E_m are centred ellipsoids. The proof is completed by setting $p = \alpha$ and using the fact that $\exp\{-\|u\|_{E_i}^\alpha\}$ is the characteristic function of a sub-Gaussian law. \square

Consider maximisation of $\mathbf{E}|\langle \xi, u \rangle|^\lambda$ over $u \in \mathbb{R}^d$ for fixed $\lambda \in (0, \alpha)$ under the constraints $\langle u, \mu \rangle = r$ for some $\mu \in \mathbb{R}_+^d$, $r \geq 0$, and $\langle u, (1, \dots, 1) \rangle = 1$. By Theorem 6.16, this is equivalent to maximising $\|u\|_F$ for u satisfying the constraints, i.e. its solution is the direction of the smallest radius-vector function for the set $F \cap H$, where

$$H = \{u \in \mathbb{R}^d : \langle u, \mu \rangle = r, \langle u, (1, \dots, 1) \rangle = 1\}.$$

This corresponds to the idea of portfolio selection studied in [2] for $\alpha > 1$.

It is also possible to consider further optimisation problems for the moments of the norm of $\eta = A\xi$, where A is an invertible linear transform and $\alpha \in [1, 2]$. The direct computation shows that η has the associated zonoid AK and the associated star body $(A^\top)^{-1}F$. By Theorem 6.1 minimising $\mathbf{E}\|A\xi\|^\lambda$ for $\lambda \in (-d, \alpha)$ over $A \in \text{SL}_n$, corresponds to the minimisation of the integral of $\|A^\top u\|_F$ over $u \in \mathbb{S}^{d-1}$.

Consider a special case of this problem for $\lambda = 1$ and $\alpha \in (1, 2]$. In terms of the associated zonoid $K = F^*$, we can equivalently minimise the mean width

$$w(AK) = 2 \int_{\mathbb{S}^{d-1}} h(AK, u) du,$$

over $A \in \text{SL}_n$. It is shown in [16] that AK has the minimal width position if the measure on the unit sphere with density $h(AK, \cdot)$ is isotropic.

Taking a subvector of ξ corresponds to a section of the associated star body F by the corresponding coordinate subspace. By applying orthogonal transformations, we see that the *projection* of ξ on any subspace H has the associated star body $(F \cap H) \oplus H^\perp$, i.e. the direct sum of $F \cap H$ and the space orthogonal to H . Therefore, the values at the origin of the probability density function of the projected ξ are closely related to the intersection body of F .

A bound on the volume of a convex set using volumes of its $(d - 1)$ -dimensional sections (see [39] and [12, p. 341]) yields the following inequality for the values of the density function at the origin

$$f(0)^{d-1} \geq \frac{\Gamma(1 + \frac{d}{\alpha})^{d-1}}{\Gamma(1 + \frac{d-1}{\alpha})^d} \frac{(d-1)!^d}{(d!)^{d-1}} \prod_{i=1}^d f_{-i}(0),$$

where $f_{-i}(0)$ is the density at the origin of the subvector of ξ with the i th coordinate excluded. This inequality holds for all $S\alpha S$ laws with convex associated star bodies. In the bivariate case,

$$f(0) \geq \frac{\Gamma(1 + \frac{2}{\alpha})}{2\Gamma(1 + \frac{1}{\alpha})^2} f_1(0) f_2(0),$$

where f_1 and f_2 are the marginal densities. Note that the coordinates of $\xi = (\xi_1, \xi_2)$ can be independent with a convex F only if $\alpha \in [1, 2]$.

Theorem 10.3. *Consider two $S\alpha S$ random vectors ξ' and ξ'' of dimensions d_1 and d_2 and the random vector $\eta = (\xi', \xi'')$ obtained by concatenating ξ' and ξ'' . Decompose \mathbb{R}^d into the direct sum of two linear subspaces H_1 and H_2 of dimensions d_1 and d_2 .*

Then ξ' and ξ'' are independent if and only if the associated star body F of η (or associated zonoid K if $\alpha \in [1, 2]$) is the α -star sum of $F_1 = (F \cap H_1) \times H_2$ and $F_2 = H_1 \times (F \cap H_2)$ (respectively K is the Firey α -sum of the projection of K onto H_1 and onto H_2 if $\alpha \in [1, 2]$).

Proof. Note that ξ' and ξ'' are independent if and only if

$$\|(u_1, u_2)\|_F^\alpha = \|u_1\|_{F_1}^\alpha + \|u_2\|_{F_2}^\alpha, \quad u_1 \in H_1, u_2 \in H_2.$$

It remains to observe that the associated star bodies of subvectors appear as the intersections of F with H_1 and H_2 and the associated zonoids are projections of K onto H_1 and H_2 respectively. \square

The duality operation transforms convex bodies into their polar sets. This operation does not generally preserve the property of a set being a zonoid or L_p -ball. However, it makes sense if applied to the spectral sets of $S\alpha S$ laws. If ξ is $S\alpha S$ with spectral set Q , then its *spectral dual* ξ^* has the spectral measure $\sigma^*(\cdot) = S_\alpha(Q^*, \cdot)$. Probabilistic studies of this operation call for geometric results concerning p th projection and centroid bodies of polar sets.

Now explore the ordering of $S\alpha S$ vectors based on inclusion relationship for their associated star bodies. Write $\eta \preceq \xi$ if $F_\xi \subset F_\eta$ for their associated star bodies F_ξ and F_η .

Theorem 10.4. *If $F_\xi \subset F_\eta$ for the associated star bodies of $S\alpha S$ random vectors ξ and η with $\alpha \in (0, 2]$, then there exist $\tilde{\xi} \stackrel{D}{=} \xi$ and $\tilde{\eta} \stackrel{D}{=} \eta$ such that $|\langle \tilde{\xi}, u \rangle| \geq |\langle \tilde{\eta}, u \rangle|$ a.s. simultaneously for all $u \in \mathbb{S}^{d-1}$.*

Proof. Fix $u \in \mathbb{S}^{d-1}$. Since $\langle \xi, u \rangle$ and $\langle \eta, u \rangle$ are $S\alpha S$ random variables with scale parameters $\|u\|_{F_\xi} \geq \|u\|_{F_\eta}$, it is possible to define ξ and η on the same probability space, so that $|\langle \tilde{\xi}, u \rangle| \geq |\langle \tilde{\eta}, u \rangle|$ a.s. By repeating the same argument, it is possible to show that finite dimensional distributions of the $|\langle \xi, u \rangle|$, $u \in \mathbb{S}^{d-1}$, are stochastically greater than the finite dimensional distributions of $|\langle \eta, u \rangle|$, $u \in \mathbb{S}^{d-1}$. The statement follows from the continuity of the processes, see also [22, Th. 4]. \square

Definition 10.5. If F_1 and F_2 are two convex sets representing the unit balls in \mathbb{R}^d with two norms, then the *Banach–Mazur distance* $\rho_{BM}(F_1, F_2)$ between F_1 and F_2 (or between the corresponding normed spaces) is defined as the infimum of $t > 0$ such that $F_1 \subset AF_2 \subset tF_1$ for an invertible matrix A , see [30, Sec. 2.1].

The *Banach–Mazur distance* $\rho_{BM}(\xi, \eta)$ between two $S\alpha S$ vectors ξ and η with $\alpha \in [1, 2]$ (needed to ensure the convexity of the associated star

bodies) is defined as the Banach–Mazur distance between their associated star bodies. By Theorem 10.4, $\rho_{BM}(\xi, \eta)$ is the infimum of $t > 0$ such that $\xi \preceq A\eta \preceq t\xi$ for an invertible matrix A . It is well known that the Banach–Mazur distance between any d -dimensional space and \mathbb{R}^d with the elliptical norm is at most \sqrt{d} . Therefore, for each $S\alpha S$ random vector ξ there exists a sub-Gaussian random vector η with the same characteristic exponent such that $\rho_{BM}(\xi, \eta) \leq \sqrt{d}$ and both η and ξ can be realised on the same probability space as ξ and $\tilde{\eta}$ so that

$$|\langle \tilde{\eta}, u \rangle| \leq |\langle A\tilde{\xi}, u \rangle| \leq \sqrt{d} |\langle \tilde{\eta}, u \rangle| \quad \text{a.s.}$$

holds simultaneously for all u . It is known that $(\mathbb{R}^d, \|\cdot\|_p)$ is the farthest from the Euclidean among all subspaces of $L_p([0, 1])$, see [30, Sec. 5.1]. Thus, the $S\alpha S$ law with independent components is the farthest one from the sub-Gaussian law with the same characteristic exponent.

Dvoretzky's theorem states that if a natural number n and $\varepsilon > 0$ are given, then every normed space of sufficiently large dimension d (depending on n and ε) has an n -dimensional subspace, whose Banach–Mazur distance from \mathbb{R}^d with an elliptical norm is less than ε . Since section of star bodies correspond to projections of $S\alpha S$ vectors, Dvoretzky's theorem implies that each $S\alpha S$ vector with convex associated star body and of sufficiently high dimension can be projected onto an n -dimensional subspace, such that its projection lies arbitrarily close to a sub-Gaussian law.

From representation (3.3) for the characteristic function one immediately obtains that if F_1, F_2, \dots is a sequence of star bodies corresponding to $S\alpha S$ vectors ξ_1, ξ_2, \dots with fixed $\alpha \in (0, 2]$, then $\xi_n \xrightarrow{d} \xi$ (converge in distribution) if and only if ξ is an $S\alpha S$ law with associated star body F satisfying $\|u\|_{F_n} \rightarrow \|u\|_F$ as $n \rightarrow \infty$ for all $u \in \mathbb{R}^d$. It is also possible to provide a version of this result for distributions from the domain of attraction of $S\alpha S$ laws.

Definition 10.6. If η is a random vector in \mathbb{R}^d , then its associated star body at level t is the star-shaped set F_t obtained by (3.2) using the spectral measure σ_t given by

$$\sigma_t(A) = \mathbf{P}\left\{\frac{\eta}{\|\eta\|} \in A \mid \|\eta\| \geq t\right\}, \quad t > 0.$$

The classical limit theorem for convergence to stable random vectors with $\alpha \in (0, 2)$ implies that if η belongs to the domain of attraction of $S\alpha S$ law ξ

if and only if $\|\eta\|$ has a regularly varying tail and σ_t converges weakly to σ being the spectral measure of ξ , see [1].

If $\alpha \geq 1$, the weak convergence of measures σ_t is equivalent to the Hausdorff convergence of the corresponding L_α -zonoids $K_t = F_t^*$. This can be proved in the same way as for $\alpha = 1$ on [48, p. 184]. For general $\alpha \in (0, 2]$, the weak convergence of σ_t to σ is equivalent to the pointwise convergence of norms $\|u\|_{F_t}$ for $u \in \mathbb{S}^{d-1}$ together with the convergence of the integrals of the norms over the unit sphere. Note that the latter convergence implies the convergence of the total masses of σ_t .

11 James orthogonality

The associated zonoid K_ξ can be used as the *scale parameter* of $S\alpha S$ random vector ξ in case $\alpha \in [1, 2]$. For general $\alpha \in (0, 2]$, the star body plays the role of the inverse scale parameter. Based on this observation, it is possible to generalise several concepts that have been defined only in the univariate and bivariate cases or for $\alpha > 1$.

The *covariation norm* $\|\eta\|_\alpha$ of $S\alpha S$ random variable η is defined to be the scale parameter of η , i.e. $\|\eta\|_\alpha = a$ if and only if $\varphi_\eta(u) = e^{-a^\alpha u^\alpha}$, see [47, Sec. 2.9]. The family \mathcal{S}_ξ of $S\alpha S$ random variables obtained as linear combinations of the coordinates of $S\alpha S$ random vector ξ in \mathbb{R}^d becomes a normed space if \mathcal{S}_ξ is equipped with the covariation norm. If ξ has the associated star body F and $\eta = \langle u, \xi \rangle$, then $\|\eta\|_\alpha = \|u\|_F$, i.e. $(\mathcal{S}_\xi, \|\cdot\|_\alpha)$ is isometric to $(\mathbb{R}^d, \|\cdot\|_F)$.

The definition of normality in normed linear spaces goes back to G. Birkhoff (1935), see [50, Sec. 3.2]. If $(X, \|\cdot\|)$ is a normed linear space, then x is normal to y (notation $x \dashv y$) if $\|x + cy\| \geq \|x\|$ for all $c \in \mathbb{R}$. This concept was later explored by R.C. James, and appeared under the name James orthogonality in the literature on stable laws.

If (ξ_1, ξ_2) are two jointly $S\alpha S$ random variables with $\alpha \in (1, 2]$, then ξ_2 is said to be *James orthogonal* to ξ_1 (notation $\xi_2 \dashv \xi_1$) if $\|c\xi_1 + \xi_2\|_\alpha \geq \|\xi_2\|_\alpha$ for all $c \in \mathbb{R}$. The James orthogonality condition can be written as

$$\|u_1\xi_1 + u_2\xi_2\|_\alpha \geq |u_2| \cdot \|\xi_2\|_\alpha, \quad u = (u_1, u_2) \in \mathbb{R}^2.$$

If $\alpha \in (1, 2]$, we have $\xi_2 \dashv \xi_1$ if and only if $[\xi_1, \xi_2]_\alpha = 0$, see [47, Prop. 2.9.2].

Theorem 11.1. *If (ξ_1, ξ_2) is $S\alpha S$ in \mathbb{R}^2 with $\alpha \in (1, 2]$ and the associated star body F , then $\xi_2 \dashv \xi_1$ if and only if $F \subset \mathbb{R} \times [-a, a]$, where $a = \rho_F((0, 1))$.*

Proof. Since the scale parameter of $(u_1\xi_1 + u_2\xi_2)$ equals $h(K, u)$, the James orthogonality condition reads $h(K, u) \geq h(K, (0, u_2))$ for all $u = (u_1, u_2) \in \mathbb{R}$. By passing to the radial function of $F = K^*$, we see that

$$\rho_F(u/\|u\|) \leq \frac{\|u\|}{|u_2|} \rho_F((0, 1)).$$

If $r(\theta) = \rho_F(\cos \theta, \sin \theta)$, then

$$r(\theta) \leq |\sin \theta|^{-1} \rho_F((0, 1)),$$

which immediately implies the statement, taking into account the equation of $\mathbb{R} \times [-a, a]$ in polar coordinates. \square

Theorem 11.1 immediately implies that independent $S\alpha S$ variables are James orthogonal and that the James orthogonality implies independence in the sub-Gaussian case, where F is an ellipsoid.

The isometry between $(S_\xi, \|\cdot\|_\alpha)$ and $(\mathbb{R}^d, \|\cdot\|_F)$ makes it possible to extend the James orthogonality concept for $\alpha \in [1, 2]$ (i.e. allow for $\alpha = 1$) and immediately yields the following result.

Theorem 11.2. *Let ξ be $S\alpha S$ with $\alpha \in [1, 2]$ and associated star body F . For each $u, v \in \mathbb{R}^d$, we have $\langle \xi, u \rangle \dashv \langle \xi, v \rangle$ if and only if $u \dashv v$ in $(\mathbb{R}^d, \|\cdot\|_F)$.*

Therefore, orthogonality property of $S\alpha S$ random variables from \mathcal{S}_ξ reduces to orthogonality in the normed space $(\mathbb{R}^d, \|\cdot\|_F)$ if F is convex. It is also possible to extend the orthogonality concept for all $\alpha \in (0, 2]$ as long as F is convex.

It is known that the orthogonality is symmetric in a normed space of dimension at least 3 if and only if the space is Euclidean, i.e. F is an ellipsoid, see [50, Th. 3.4.10]. The corresponding probabilistic result is a part of [47, Prop. 2.9.3]. In dimension $d = 2$ the orthogonality is symmetric if and only if the boundary of F is a *Radon curve*, see [6] and [50, p. 94]. The corresponding question for $S\alpha S$ laws was posed as an open problem in [47, p. 109]. Recall that ∂F is a Radon curve if and only if the boundary of F in the second and fourth quadrants coincides with the boundary of the projection body of $K = F^*$.

The James orthogonality is a property of the associated star body or associated zonoid of an $S\alpha S$ law and is not directly influenced by α . If it holds for an $S\alpha S$ law, then it applies for all symmetric stable laws that share the same associated star body.

It is also possible to define multivariate extensions of the James orthogonality concept.

Definition 11.3. If ξ and η are $S\alpha S$ in \mathbb{R}^d with $\alpha \in [1, 2]$, then η is said to be

- (i) James orthogonal to ξ (notation $\eta \dashv \xi$) if the associated zonoid of $c\xi + \eta$ contains the associated zonoid of η for all $c \in \mathbb{R}$;
- (ii) strongly James orthogonal to ξ (notation $\eta \dashv_S \xi$) if $\langle v, \eta \rangle$ is James orthogonal to $\langle u, \xi \rangle$ for all $u, v \in \mathbb{R}^d$.

The strong James orthogonality is linear invariant, i.e. all linear transformations preserve this property. It is easy to see that if $\eta \dashv_S \xi$, then the associated zonoid of $(c\xi + \eta)$ contains the associated zonoid of η for all $c \in \mathbb{R}$, i.e. the strong orthogonality implies (i). For this it suffices to note that this associated zonoid has the support function $h(K, (cx, x))$ and apply Definition 11.3(ii) with $u = cx$ and $v = x$.

Theorem 11.4. If (ξ, η) is $S\alpha S$ in \mathbb{R}^{2d} with $\alpha \in [1, 2]$ and the associated star body F , then $\eta \dashv_S \xi$ if and only if

$$\|u + v\|_F \geq \|v\|_F \quad (11.1)$$

for all $u = (u_1, \dots, u_d, 0, \dots, 0)$ and $v = (0, \dots, 0, v_1, \dots, v_d)$. Furthermore, $\eta \dashv \xi$ if and only if (11.1) holds for $u = (c, \dots, c, 0, \dots, 0)$, $v = (0, \dots, 0, 1, \dots, 1)$ and all $c \in \mathbb{R}$.

Proof. For each $u', u'' \in \mathbb{R}^d$, the scale parameter of $c\langle u', \xi \rangle + \langle u'', \eta \rangle$ is $\|(cu', u'')\|_F$. By the condition, this is at least $\|(0, u'')\|_F$, which is the scale parameter of $\langle u'', \eta \rangle$. \square

If ξ and η from Theorem 11.4 are independent, then F is the α -star sum of the associated star bodies of ξ and η , i.e.

$$\|u + v\|_F^\alpha = \|u + v\|_{F_\xi}^\alpha + \|u + v\|_{F_\eta}^\alpha = \|u\|_{F_\xi}^\alpha + \|v\|_{F_\eta}^\alpha \geq \|v\|_{F_\eta}^\alpha,$$

i.e. η is strong James orthogonal to ξ .

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